ON CLASSIFICATION OF VARIATIONAL CONSERVATION LAWS AND SOLUTIONS OF DAMPED WAVE EQUARIONS IN CERTAIN RIEMANNIAN METRICS

BY

USAMAH SADEG AL-ALI

A Dissertation Presented to the

DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICS

February 2017

On Classification Of Variational Conservation Laws and Solutions of Damped Wave Equations in Certain Riemannian Metrics

By

Usamah Sadeg Al-Ali

A dissertation presented to the

Deanship of Graduate Studies

In Partial Fulfilment of the Requirements

for the Degree

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

KING FAHD UNIVERSITY

OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

November, 2016

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN- 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This dissertation, written by **USAMAH SADEG AL-ALI** under the direction of his dissertation advisor and approved by his dissertation committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY OF SCIENCE IN MATHEMATICS.**

Dissertation Committee

Prof. Ashfaque H. Bokhari
(Dissertation Advisor)

Prof. F. D. Zaman
(Co-Advisor)

Prof. Hassan Azad
(Member)

Prof. Hocine Bahlouli
(Member)

Prof. Ahmet Sahin
(Member)

Dr. Hussein S. Al-Attas
Department Chairman

Dr. Salam A. Zummo
Dean of Graduate Studies

5/4/17
Date

To the memory of my parents

ACKNOWLEDGEMENTS

First of all, my gratitude goes to ALLAH for his guidance and protection in every stage of my life. I am extremely grateful to my advisor Professor Ashfaque Bokhari for his unconditional support throughout this project. I would like to express my sincere appreciation to him for taking time out of his busy schedule to provide his expertise that greatly assisted in the completion of this work.
My gratitude also extends to my dissertation committee members, Professor Fiaz Zaman, Professor Hassan Azad, Professor Hocine Bahlouli and professor Ahmet Sahin for their valuable contribution to reviewing this work. My thankfulness should be expressed for Dr Tahir Mustafa, the one who taught me the subject of symmetry few years ago, which made it possible for me to work in this fascinating field. Thanks must be given to professor Abdul Hamid Kara and Dr. Ahmed Al-Dweik as well for their constructive criticism and fruitful suggestions that led to improvement of the project. I would like also to extend my appreciation to my colleague Mr Jawad Al-Ali for his unforgettable assistance which made it possible for me to maintain a balance between research duties and teaching responsibilities .
Finally, this dissertation could not have been written without the support from KFUPM , and especially the math department, which enabled me to finalize this project within the limited time frame.

CONTENTS

LIST OF TABLES

LIST OF FIGURES

DISSERTATION ABSTRACT

NAME: Usamah Sadeg Al-Ali

TITLE OF STUDY: On Classification Of Variational Conservation Laws and Solutions of Damped Wave Equations in Certain Riemannian Metrics

MAJOR FIELD: Mathematics

DATE OF DEGREE: February, 2017

In this dissertation, our focus is to solve some evolution type equations of interest in mathematical physics, in both canonical as well as in Lorentzian geometries. To investigate solutions and other physical properties admitted by the considered equations, we mainly adhere to using the Lie symmetry and Noether symmetry approach for partial differential equations. The dissertation comprises of seven chapters, of which the first two chapters are devoted to a brief discussion of preliminaries and basic notions used in the research carried out in the later chapters.

After a brief discussion of preliminaries, we have focused our attention on finding a complete classification of a plane symmetric non-static spacetime metric by its variational symmetries. It is shown that the number of variational symmetries for the studied spacetime metric ranges between '4' and '17', respectively representing minimal and maximal symmetry algebra of the variational symmetries.

At this stage, we digress to study two damped wave equations in one and higher dimensions. Exploiting the well-known Lie symmetry method, we first carry out a symmetry classification of a one dimensional damped wave equation with a damping force and a source term, along with a (2+1) damped wave equations with variable damping. By using the derived symmetries in both cases, certain solutions of the damped equations are presented.

Keeping in view the importance of propagation of disturbances on the surface of locally curved objects, we have considered the damped wave equation on the surface of the sphere. Using the derived Lie symmetries of the equation, we perform certain similarity reductions of the considered equation. In each case, the studied damped wave equation is transformed to an ordinary differential equation (ODE),and later exact solutions are obtained by a further analysis of these ODEs.

Finally, we have also focussed on an interesting problem by considering the relativistic heat equations in Friedmann-Lematre-Robertson-Walker flat spacetime metric. This is a problem of interest in standard cosmology as being a solution of the Einstein field equations. Using the separation of variable method in this case, we have obtained exact solutions of the Relativistic heat equation corresponding to radiation, matter and dark energy dominated universes. Interestingly, we have found that when the dark energy completely dominates the contents of the universe and the expansion is purely exponential (which is not yet the case), a solution obtained represents decrease of the cosmic microwave background temperature . For completeness, we also employ the Lie symmetry method to find Lie symmetries of the considered equation and perform certain reductions.

ملخص الرسالة

الاسم : اسامة صادق العلي

عنوان الرسالة : تصنيف قوانين الحفظ التغيرية و حلول المعادلات الموجية المثبطة في دوال مترية ريمانية معينة

التخصص : رياضيات

تاريخ التخرج : فبراير 2017

في هذا البحث، نركز اهتمامانا على حل بعض معادلات التطور المهمة في الفيزياء الرياضية في كل من الهندسات المعيارية و اللورنتزية. من اجل البحث عن الحلول و الخواص الفيزيائية الاخرى التي تنطبق على هذه المعادلات، نلتزم بشكل رئيسي بتطبيق المقاربات التي تعتمد على تناظرات "لي" و تناظرات "نويثر" للمعادلات التفاضلية الجزئية. تتكون الاطروحة من سبعة فصول حيث تم تخصيص الفصلين الاول و الثاني لمناقشة مختصرة للمفاهيم الاساسية التي سوف تستخدم خلال الفصول اللاحقة من هذا البحث.

بعد عرض المفاهيم الاساسية، سنقوم بإجراء تصنيفا تماثليا كاملا للزمكانات غير الثابتة والمتناظرة على محورين من حيث التناظرات التغيرية التي تسمح بها هذه الزمكانات. لقد قمنا بإثبات ان عدد التناظرات التغيرية لهذا النوع من الزمكانات قيد الدراسة يتراوح بين 4 و 17 تناظرا تمثل تناظرات الجبر الادنى و تناظرات الجبر الاعلى على التوالي.

بعد ذلك نقوم بدراسة نوعين من المعادلات الموجية المثبطة في بعد واحد و ابعاد متعددة. باستخدام تناظرات " لي "سنقوم بإجراء تصنيفا تماثليا للمعادلة الموجية المثبطة احادية البعد مع قوة مثبطة و حد اضافي يمثل المصدر، بجانب معادلة اخرى ذات ابعاد (2+1) مع قوة مثبطة متغيرة. باستخدام التناظرات المستنتجة سنوجد حلولا للمعادلتين قيد الدراسة.

واضعين نصب اعيننا اهمية دراسة التذبذبات الموجية على الاجسام المنحنية محليا، قمنا بدراسة المعادلة الموجية المثبطة على سطح الكرة. باستخدام تناظرات " لي " المستخلصة سنقوم باختزال المعادلة الموجية قيد الدراسة، حيث سيتم تقليصها في كل حالة اختزال الى معادلة تفاضلية عادية و من ثم ايجاد حلول لها عن طريق تحليل المعادلات المختزلة.

اخيرا، سوف نركز اهتمامنا على دراسة معادلة الحرارة النسبوية في زمكان فريدمان – لاميتير-روبرتسون - وولكر المسطح. هذه المسالة مهمة في علم الكونيات حيث انها تمثل حلولا لمعادلات المجال لأينشتين. باستخدام طريقة فصل المتغيرات ، قمنا باشتقاق حلولا دقيقة لمعادلة الحرارة قيد الدراسة تمثل نماذج مختلفة للكون، تحديدا نموذج الكون المحكوم بالطاقة، والكون المحكوم بالمادة، واخيرا الكون المحكوم بالطاقة المظلمة. لقد توصلنا في هذا البحث لاستنتاج مثير حيث وجدنا انه عندما تهيمن الطاقة المظلمة على الكون بشكل مطلق و يصبح التمدد الكوني ذات طابع اسي (وهي الحالة التي لم تحدث بعد) فان الحلول التي اوجدناها تمثل انخفاضا في درجة حرارة اشعة الخلفية الكونية. من اجل تحقيق الشمولية في هذه الجزئية من البحث، قمنا بتحليل الخواص التناظرية لمعادلة الحرارة النسبوية قيد الدراسة لاستخدام هذه التناظرات في إجراء اختزالات معينة لهذه المعادلة.

INTRODUCTION

Symmetry plays a pivotal role in sciences as it underlies some of the most insightful results found in many fields. In fact, physical systems are better understood by investigating their symmetry properties. In this context, the variational symmetries, also known as Noether symmetries, are of considerable interest as they attracted the attention of researchers since the emergence of Noether theorem in 1918. In the seminal work of Emmy Noether, she introduced a very powerful theorem, which led to a deeper understanding of the connection between conservation laws and symmetries. According to Noether's theorem, for every continuous symmetry admitted by the Lagrangian of a given system, there exists a conservation law. This means that, in principle, one obtains the conservation laws governing a physical system from its observed symmetries.

On the other hand, the well-known Lie point symmetries are of great interest too as they have proved to be powerful tools in solving non-linear differential equations if they admit them. These Lie symmetries have been extensively studied and implemented to solve differential equations where the classical techniques fail to find their exact solutions. In this context, the Lie symmetry method has been implemented in investigating various classes of evolution type equations in which wave and heat equations have gained a considerable attention.

In fact, investigating the formation and motion of waves constitutes a significant part of the studies in applied mathematics. In this regard, study of wave propagation has remained a major area of investigation in areas such as acoustics, hydrodynamics [24], optics [25], electromagnetism, general relativity[6], and classical and quantum mechanics. Nevertheless, classical wave model lacks the ability to predict the behavior of wave phenomena arising in physical systems under certain circumstances. Such situations are frequently met when dealing with waves whose behavior exhibit diffusion phenomena via the damped wave equation(s).

A damped wave in general is a wave whose amplitude of oscillation decreases with time. This damping is expressed mathematically by including a damping term in the classical wave equation. The damping term has the effect of controlling the speed of oscillation. Thus, the damped equation model can describe effectively many physical systems involving processes that dissipate the energy stored in the oscillation. Whereas a lot of work has appeared on Lie point symmetries for wave and other classes of evolution type equations, the damped wave equations have so far been mostly left un-addressed. Therefore, addressing the damped wave equations is essential in understanding the wave behavior in many physical systems of interest.

In this dissertation, we intend to implement both types of symmetries, the Noether symmetries and the Lie symmetries, to study some fundamental equations of mathematical physics in both classical and certain Riemannian metrics of interest. A brief outline of the research presented in this dissertation pivots around the following:

In the introductory part (first two chapters), we introduce the basic notions, definitions and theorems that are necessary to understand the subjects being discussed in this dissertation. In particular, in the first chapter we briefly introduce Riemannian manifolds where attention is focused on Lorentzian manifolds and Lorentzian metrics. In the second chapter, we briefly present an outline of the Noether two theorems by providing a simple example on how Noether theorem is used to construct conservation laws. Additionally, a detailed description of the procedure of obtaining Noether symmetries is also given.

In the remaining five chapters, we address five interesting problems. In particular, in chapter three, we carry out a complete classification of plane symmetric non-static spacetime of the form,

$$ds^2 = e^{\nu(t,z)}dt^2 - e^{\lambda(t,z)}[dx^2 + dy^2] - e^{\mu(t,z)}dz^2,$$

in terms of Noether symmetries admitted by them. The derived Noether symmetries are also utilized to construct conservation laws for the general form of the above metric. The obtained results are also applied to some interesting plane symmetric spacetime metrics arising in general relativity.

In the fourth and fifth chapters, we present a group classification of two different types of damped wave equations. In this regard, the one dimensional damped wave equation with a damping force and a source term of the form,

$$u_{tt} + f(u)u_t = u_{xx} + g(u),$$

is studied in chapter four. Furthermore, the (2+1) damped wave equation of the form,

$$u_{tt} + f(u)u_t = div(g(u)grad\ u),$$

is investigated in chapter five. In both chapters, some solutions of the considered equations are obtained by utilizing their derived symmetries.

In chapter six, we construct the damped wave equation of the form,

$$u_{tt} + \alpha u_t = \triangle u,$$

on the surface of a sphere. The Lie symmetries of the equation are obtained and then implemented in reducing the equation into an ODE, where exact solutions are found by further investigating the reduced equations.

In the last chapter, the method of separation of variables is utilized to study the relativistic heat equation,

$$\beta u_t = \Box u,$$

in the flat Friedmann spacetime, $ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$. Solutions of the considered equation are derived corresponding to radiation, matter and dark energy dominated universes. Moreover, boundary conditions are considered for the model of dark energy dominated universe. These boundary conditions give rise to an exact solution that enables one to trace the temperature evolution in our actual universe. The investigation of the relativistic heat equation is further extended to its symmetry structure, where we use its symmetries to reduce the equation into a second order PDE of two independent variables.

1. PRELIMINARIES

In this chapter, we present the fundamental notions of Riemannian and pseudo-Riemannian geometries used in this dissertation. In particular, the concept of a differentiable manifold is introduced in section 1.1 and the definition of a Riemannian manifold in section 1.2. In section 1.3, we focus our attention on spacetime metrics in general and in particular the Killing vectors are discussed in section 1.4. We conclude the chapter by giving a brief discussion of the relationship between conservation laws and the Killing vectors.

1.1 ***Differentiable Manifolds***

The geometric properties of Euclidean space were well-understood long ago. However, efforts to study and deal with structures that are more complicated gave rise to the notion of a manifold which later became central in the study of geometry, mathematical physics and General Relativity.

Definition 1.1.1. A topological space M is called a manifold of dimension m if every point $p \in M$ has a neighborhood $U \subset M$ that is homeomorphic to some open subset of R^m.

Notice that the definition 1.1.1 implies that the manifold in general is a topological space that locally resembles Euclidean space near each point of its domain. The collections of homeomorphisms $\varphi : U \to R^m$ is called a coordinate chart. The coordinate charts are necessary in the construction of atlases on the manifold.

Definition 1.1.2. An atlas of a manifold M is a collection (U_a, ϕ_a) of charts such that $\bigcup U_a = M$.

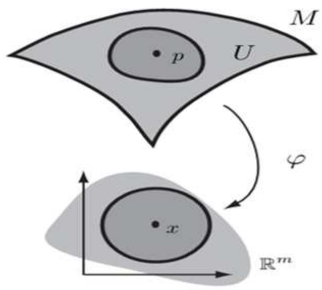

Fig. 1.1: Homeomorphisms between neighborhood of p and subset of R^m

Definition 1.1.3. Suppose that (U_a, φ_a) and (U_b, φ_b) are two charts of a manifold M such that $U_a \bigcap U_b$ is not empty then the transition map,

$$T_{a,b} : \varphi_a(U_a \bigcap U_b) \to \varphi_b(U_a \bigcap U_b), \tag{1.1.1}$$

is defined by $T_{a,b} = \varphi_b \; o \; \varphi_a^{-1}$.

In general, we can not do calculus on a manifold unless it satisfies additional property. This additional property gives rise to the notion of differentiable manifolds, which play central role in differential geometry.

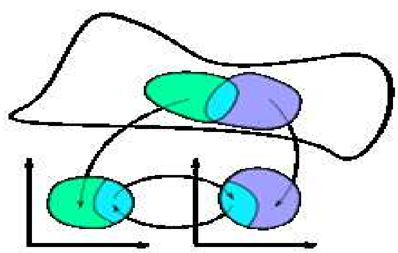

Fig. 1.2: Two charts on a manifold

Definition 1.1.4. A topological manifold M is said to be differentiable if the transition maps $T_{a,b} = \varphi_b \ o \ \varphi_a^{-1}$ for all pairs φ_a, φ_b in the atlas are diffeomorphisms.

There are many familiar examples of differentiable manifolds. Figure 1.3 below shows some of these familiar examples:

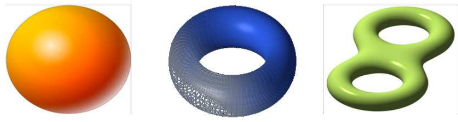

Fig. 1.3: Some examples of differentiable manifolds: sphere, torus and double torus.

1.2 Riemannian manifolds

As mentioned above, the differentiable manifolds are central objects in differential geometry. In particular, if we want to define the length of vectors or the angle between two vectors in a given differentiable manifold, then we are naturally led to the notion of a Riemannian manifold and a metric associated with it.

Definition 1.2.1. A Riemannian manifold is a differentiable manifold in which each tangent space is equipped with an inner product $< .,. >$.

The local geometry of Riemannian manifolds can be described by defining the distance (interval) between two points as,

$$ds^2 = g_{ab} dx^a dx^b, \tag{1.2.1}$$

where g_{ab} represents the components of Riemannian metric.

Definition 1.2.2. Let M be a differentiable manifold of dimension m. A

Riemannian metric on M is a family of (positive definite) inner products,

$$g_p : T_pM \times T_pM \to R, \qquad p \in M \tag{1.2.2}$$

such that, for all differentiable vector fields X, Y on M, $p \to g_p(X(p), Y(p))$ defines a smooth function $M \to R$.

If the condition of positive definiteness in above definition is eliminated, it gives rise to the so-called pseudo-Riemannian manifolds which are generalization of Riemannian manifolds.

Definition 1.2.3. A pseudo-Riemannian manifold is a generalization of a Riemannian manifold in which the metric tensor need not be positive definite.

The Riemannian and pseudo-Riemannian manifolds are discussed in many references, see for example [1–3]. The pseudo-Riemannian manifolds are extremely important as they play a pivotal role in the study of curved surfaces. In particular, a significant subclass of pseudo-Riemannian manifolds is the Lorentzian manifolds which have wide applications in General Relativity.

Definition 1.2.4. A Lorentzian manifold is a subspace of pseudo-Riemannian manifold which has a signature $+2$ or -2.

The significance of Lorentzian manifolds in General Relativity arises from the fact that spacetime can be modeled as a 4-dimensional manifold of signature (3,1) or equivalently (1,3). For example, the well-known Minkowski spacetime can be written as a signature -2 metric:

$$ds^2 = g_{ab}dx^a dx^b = dt^2 - dx^2 - dy^2 - dz^2, \qquad a, b = 0, ..., 3. \tag{1.2.3}$$

Equivalently, it can also be written as a signature $+2$ metric:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{1.2.4}$$

1.3 Spacetimes

The notion of a spacetime in mathematics refers to any mathematical model that combines space and time into a single continuum, equation (1.2.3) and (1.2.4) are two examples. This notion was developed as a direct result of the discovery of the Einstein theory of Relativity in the beginning of the preceding century. In fact, combining space and time into a single indivisible spacetime proved to be a very powerful tool in simplifying many physical problems and consequently lead to a deeper understanding of the Einstein theory of Relativity and other modern theories of mathematical physics. See [4–6] for more explanation of topics related to spacetime geometry and relativity.

Definition 1.3.1. A spacetime is a four dimensional smooth connected Lorentzian manifold (M, g) of signature (3,1) or (1,3).

Notice that the metric of a spacetime g determines its geometry. In particular a metric in additional to other important physical applications, is used to determine the geodesics of particles and light beams. The points of the spacetime manifolds are events. One of the most powerful properties of spacetime geometry arises via the symmetries they admit.

Definition 1.3.2. A symmetry of a spacetime is a vector field,

$$X = \eta^i \frac{\partial}{\partial x^i},$$

whose local flow diffeomorphisms preserve some property of the spacetime.

Actually, implementing symmetries of spacetime is pivotal in studying many physical problems arising in general relativity and cosmology. For example, the German mathematician Schwarzschild was able to obtain the first exact solution to the Einstein field equations of General Relativity by exploiting spherical symmetry via investigating the curvature produced by a massive nonrotating spherical object in an empty space [7].

As far as GR is concerned, spacetime symmetries usually require some form of preserving properties. The preserving properties, for instance, include preserving, the geodesics of the spacetime, the metric and the curvature tensors. In particular, isometries are the spacetime symmetries that preserve the metric itself. The infinitesimals of isometries are called Killing vector fields.

1.4 ***Killing Vector Fields***

A Killing vector field is one of the most significant and interesting spacetime symmetries. Killing preserve the metric tensor as it maintains distances in the sense that moving any set of points on a Riemannian manifold in the direction of a Killing vector field will keep the distance relationships on the manifold unchanged. This fact has an important implication in General Relativity as it indicates that spacetime metric does not change in the direction of a vector field.

Definition 1.4.1. A vector field X on a pseudo-Riemannian manifold M is said to be a Killing field if the Lie derivative with respect to X of the metric on this manifold vanishes, i.e

$$Ł_X g = 0. \tag{1.4.1}$$

Recall that the Lie derivative evaluates the change of a tensor field along the flow of another vector field. In particular, if T_{ab} is a tensor field, then its Lie derivative with respect to a vector field X is defined as,

$$L_X T_{ab} = \lim_{\epsilon \to 0} \frac{T'_{ab}(x') - T_{ab}(x)}{\epsilon} \ . \tag{1.4.2}$$

Equivalently, an explicit formula for the Lie derivative, in a torsion free space, can be written as,

$$L_X T_{ab} = T_{ad} X^d_{,b} + T_{db} X^d_{,a} + T_{ab,e} X^e. \tag{1.4.3}$$

A detailed introduction to isometry and Killing vector fields is available in [3].

Example 1.4.1.

We can construct the Killing vector fields of the two dimensional Euclidean space by applying the Lie equation (1.4.1). Recall that the metric of the xy-plane is given by,

$$ds^2 = dx^2 + dy^2. \tag{1.4.4}$$

Assume the Killing symmetry is of the form,

$$X = \xi^1(x,y)\frac{\partial}{\partial x} + \xi^2(x,y)\frac{\partial}{\partial y} \ . \tag{1.4.5}$$

Let K_{ab} be the components of the Lie derivative of the metric (1.4.4) with respect to the vector field (1.4.5), then equation (1.4.3) implies the following:

$$K_{ab} = L_X g_{ab} = g_{ad} X^d_{,b} + g_{db} X^d_{,a} + g_{ab,e} X^e. \tag{1.4.6}$$

Thus, the components K_{ab} are obtained as follows:

$$K_{11} = g_{1d} X^d_{,1} + g_{d1} X^d_{,1} + g_{11,e} X^e$$

$$= g_{11} X^1_{,1} + g_{11} X^1_{,1} + 0 = 2X^1_{,1} = 2\xi^1_x,$$

$$K_{12} = g_{1d} X^d_{,2} + g_{d2} X^d_{,1} + g_{12,e} X^e$$

$$= g_{11} X^1_{,2} + g_{22} X^2_{,1} + 0 = X^1_{,2} + X^2_{,1} = \xi^1_y + \xi^2_x,$$

$$K_{21} = g_{2d}X^d_{,1} + g_{d1}X^d_{,2} + g_{21,e}X^e$$

$$= g_{22}X^2_{,1} + g_{11}X^1_{,2} + 0 = X^2_{,1} + X^1_{,2} = \xi^2_x + \xi^1_y,$$

$$K_{22} = g_{2d}X^d_{,2} + g_{d2}X^d_{,2} + g_{22,e}X^e$$

$$= g_{22}X^2_{,2} + g_{22}X^2_{,2} + 0 = X^2_{,2} + X^2_{,2} = 2X^2_{,2} = 2\xi^2_y.$$

Now, we substitute the obtained components of K_{ab} in Lie equation (1.4.1) to obtain the following system of differential equations:

$$\xi^1_x = 0, \tag{1.4.7}$$

$$\xi^2_x + \xi^1_y = 0, \tag{1.4.8}$$

$$\xi^2_y = 0, \tag{1.4.9}$$

for conventional reason we represent the components of the vector X as: ξ^1, ξ^2. The above system of coupled equations (1.4.7) to (1.4.9) can be easily solved to obtain their solution given by,

$$\xi^1 = c_2 - c_1 y, \qquad \xi^2 = c_3 + c_1 x. \tag{1.4.10}$$

The above solution gives rise to three Killing symmetries, where each constant in above solution corresponds to a Killing field. These Killing vectors can be constructed from Eq.(1.4.5) as follows:

$$X_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

$$X_2 = \frac{\partial}{\partial x},$$

$$X_3 = \frac{\partial}{\partial y}.$$

The significance of Killing vector fields arises from the fact that they can be used to construct conserved currents. More generally, a Killing vector field gives rise to a conservation law via the Noether theorem, which will be discussed in details in the next chapter.

1.5 ***Conservation Laws***

Conservation laws are fundamental to one's understanding of the physical world. In fact, they are of particular interest in mechanics. Uncovering conserved quantities in a given physical system is essential in understanding the behavior of that system. For instance, if the conserved quantities have physical meaning such as mass, energy, momentum, charge or other constant of motion, then conservation laws are used to describe the dynamics of the system. On the other hand, if the conserved quantities do not describe any physical property of the system, they are still important in that they can be used to give the integrability of the system.

In general, any law stating that a particular measurable property of an isolated physical system does not change as the system evolves over time, is called a conservation law.

Definition 1.5.1. A function $F(t, q, \dot{q}, \ddot{q}, ..., q^{(n-1)})$ is called a constant of motion or a conserved quantity if its total time derivative satisfies the property that:

$$D_t F(t, q, \dot{q}, \ddot{q}, ..., q^{(n-1)}) = 0,$$

or equivalently,

$$\frac{\partial F}{\partial t} + \sum_i \big(\frac{\partial F}{\partial q_i}\dot{q_i} + \frac{\partial F}{\partial \dot{q_i}}\ddot{q_i} + ... \frac{\partial F}{\partial q_i^{(n-1)}} q_i^{(n)}\big) = 0. \tag{1.5.1}$$

2. NOETHER THEOREM

Noether theorem provides one of the most profound and elegant relations in modern physics. The pioneering work of Noether gave an insight into the relationship between conservation laws and symmetries. Since Noether theorem was based on the Lagrangian formulation, we start this chapter with a brief introduction of Lagrangian. In section 2.2, we give the first Noether theorem. The procedure for obtaining Noether symmetries is provided in section 2.3. The second Noether theorem is presented in section 2.4 where we provide an example by constructing conservation laws using the theorem.

2.1 ***Introduction to Lagrangian Mechanics***

The Lagrangian mechanics is an alternative approach to formulate the laws of motion. This alternative formulation is equivalent to Newtonian formulation but it has the advantage of being easier to implement in most practical applications. The main reason for the simplicity of the Lagrangian approach is that the Lagrangian mechanics is expressed in terms of energies, while Newtonian mechanics is expressed in terms of forces. This means that Lagrangian mechanics deals with scalar quantities , energy, whereas Newtonian mechanics deals with vector quantities.

Definition 2.1.1. The Lagrangian is defined to be a function of position $q^A(t)$

and the velocities $q'^A(t)$ of particles, given by,

$$L(t, q^A(t), q'^A(t)) = T(q'^A) - V(q^A), \tag{2.1.1}$$

where $V(r)$ and T are potential and kinetic energies respectively with T,

$$T = \frac{1}{2}\sum_A m_A(q'^A)^2.$$

Notice that the Lagrangian in general may depend on several functions q_j's of several independent variables t_i's with higher derivatives $q''_j, q'''_j, ..., q_j^{(n)}$. However, we restrict ourselves to Lagrangian depending on one independent variable t.

Definition 2.1.2. To each path in a system with fixed endpoints, we assign an action defined as,

$$S = \int_{t_i}^{t_f} L(t, q_j(t), q'_j(t), ..., q_j{}^{(n)}(t))dt. \tag{2.1.2}$$

One of the fundamental principle in Physics is called the principle of stationarity of action. According to this principle, the actual trajectory taken by a system is the one that yields a stationary value of the action. This principle gives rise to a significant differential equation used to derive the equations of motion for any physical system possessing a Lagrangian. These equations, called Euler-Lagrange equations, given by:

$$\frac{\partial L}{\partial q_j} + \sum_{i=1}^{n}(-1)^i \frac{d^i}{dt^i}\left(\frac{\partial L}{\partial q_j^{(i)}}\right) = 0. \tag{2.1.3}$$

2.2 First Noether Theorem

Theorem 2.2.1. For each a one parameter family of continuous maps $x_i(t) \to \overline{x_i}(t, \varepsilon)$ for the Lagrangian $L(x, \dot{x}, t)$ where $\overline{x_i}(t, 0) = x_i(t)$, there corresponds a conserved quantity.

Proof

$$\frac{\partial L(x_i, \dot{x_i}, \varepsilon)}{\partial \varepsilon} = \frac{\partial L}{\partial \overline{x_i}} \frac{\partial \overline{x_i}}{\partial \varepsilon} + \frac{\partial L}{\partial \overline{\dot{x_i}}} \frac{\partial \overline{\dot{x_i}}}{\partial \varepsilon} \tag{2.2.1}$$

$$0 = \frac{\partial L(x_i, \dot{x_i}, \varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \frac{\partial L}{\partial \overline{x_i}} \frac{\partial \overline{x_i}}{\partial \varepsilon}\bigg|_{\varepsilon=0} + \frac{\partial L}{\partial \overline{\dot{x_i}}} \frac{\partial \overline{\dot{x_i}}}{\partial \varepsilon}\bigg|_{\varepsilon=0} \tag{2.2.2}$$

$$= \frac{\partial L}{\partial x_i} \frac{\partial \overline{x_i}}{\partial \varepsilon}\bigg|_{\varepsilon=0} + \frac{\partial L}{\partial \dot{x_i}} \frac{\partial \overline{\dot{x_i}}}{\partial \varepsilon}\bigg|_{\varepsilon=0} \tag{2.2.3}$$

$$= \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) \frac{\partial \overline{x_i}}{\partial \varepsilon}\bigg|_{\varepsilon=0} + \frac{\partial L}{\partial \dot{x_i}} \frac{\partial \overline{\dot{x_i}}}{\partial \varepsilon}\bigg|_{\varepsilon=0} \tag{2.2.4}$$

$$= \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}} \frac{\partial \overline{x_i}}{\partial \varepsilon}\Big)\bigg|_{\varepsilon=0}. \tag{2.2.5}$$

Hence, the quantity $\sum_i \frac{\partial L}{\partial \dot{x}} \frac{\partial \overline{x_i}}{\partial \varepsilon}$ at $\varepsilon = 0$ is conserved.

Example 2.2.1

If the Lagrangian L is invariant under translation in time i.e. $\frac{\partial L}{\partial t} = 0$ then the total energy of the system represented by the Hamiltonian

$$H = \sum_i \dot{x_i} \frac{\partial L}{\partial \dot{x_i}} - L, \tag{2.2.6}$$

is conserved. This result can be easily verified by differentiating the Hamiltonian H with respect to t as shown below:

$$D_t H = D_t\Big(\sum_i \dot{x_i} \frac{\partial L}{\partial \dot{x_i}} - L\Big) \tag{2.2.7}$$

$$= \sum_i [\ddot{x_i}\frac{\partial L}{\partial \dot{x_i}} + \dot{x_i}\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \dot{x_i}\frac{\partial L}{\partial x_i} - \ddot{x_i}(\frac{\partial L}{\partial \dot{x_i}})] - \frac{\partial L}{\partial t} \tag{2.2.8}$$

$$= \sum_i \dot{x_i}[\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x_i}] \tag{2.2.9}$$

$$= 0. \tag{2.2.10}$$

The example 2.2.1 shows that the homogeneity of time in a physical system gives rise to conservation of energy. Similarly, the Noether's theorem predicts that homogeneity of space leads to the conservation of linear momentum as we will see in the next example.

Example 2.2.2

Consider a free particle of mass m and velocity v moving along a straight line in Euclidean space. In the absence of potential energy, the Lagrangian is equal to the kinetic energy only, i.e. $L = \frac{1}{2}m\dot{x}^{\,2}$. Obviously, the Lagrangian in this case is invariant under translation in space as $\frac{\partial L}{\partial x} = 0$. Therefore, the equation of motion for this particle, using the Euler -Lagrange equation (2.1.3), takes the form:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = 0 \tag{2.2.11}$$

$$\frac{d}{dt}(\ m\dot{x}\ ) = 0 \tag{2.2.12}$$

$$\frac{d}{dt}P = 0. \tag{2.2.13}$$

The above equation states that the linear momentum P is conserved. In a similar manner, one can show that the isotropy of space gives rise to the conservation of angular momentum, that is, if the Lagrangian is invariant under rotation then the total angular momentum of the system remains unchanged. For more details on Noether theorem and conservation laws, we refer to [8–10].

2.3 Noether Symmetries

Noether symmetries of a given physical system are the invariants of its Lagrangian up to a gauge term. In this section, we restrict ourselves to Lagrangians that depend on the coordinate x, the velocity $\dot{x}$ and time t but the definitions can be generalized to any Lagrangian.

Definition 2.3.1 A vector field of the form,

$$X = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x} \ , \tag{2.3.1}$$

is a Noether symmetry if it leaves the Lagrangian $L(t, x, \dot{x})$ invariant.

Although the above definition tells what a Noether symmetry is, it is not helpful if one wants to derive Noether symmetries for a given physical system. In order to obtain Noether symmetries for any system, we can apply the following invariance criterion.

Definition 2.3.2 A vector field of the form (2.3.1) is a Noether symmetry if it satisfies the following equation:

$$X^{[n]}L + LD_t\xi = D_t f(t,x), \tag{2.3.2}$$

where X^n is the n'th prolongation of the symmetry generator X, n is the order of the highest derivative of L, f is the gauge term and D_t is the differential operator defined by,

$$D_t = \frac{\partial}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} \ . \tag{2.3.3}$$

Recall that the prolongation $X^{[n]}$ is an extension of the infinitesimal symmetry generator X that was implemented to include the transformations of all dependent variables of the considered system along with their derivatives. This extension is given by the formula,

$$X^{[n]} = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x} + \sum_{k=1}^{n} \eta^{[k]}\frac{\partial}{\partial x^{(k)}} \ . \tag{2.3.4}$$

The coefficients $\eta^{[k]}$ are given by the formula,

$$\eta^{[k]} = \frac{d\eta^{k-1}}{dt} - x^{(k)}\frac{d\xi}{dt} \ . \tag{2.3.5}$$

For details we refer the reader to [11]. For completeness, we list below steps involved for finding Noether symmetries:

Procedure for Obtaining Noether Symmetries

Step 1. Apply the invariance criterion (2.3.2).

Step 2. Compare the coefficients of the derivatives of the dependent variable in step 1 to obtain the determining equations.

Step 3. Solve the determining equations.

Step 4. Construct the Noether symmetry corresponding to each constant in the derived solution of the determining system by substituting in the Noether symmetry generator (2.3.1).

To show the procedure for obtaining Noether symmetries, we introduce the following example.

Example 2.3.1

A particle of mass m submitted to a nonconservative force proportional to the square of its velocity according to the equation,

$$\ddot{x} + \alpha \dot{x}^{\,2} = 0. \tag{2.3.6}$$

A lagrangian of Eq.(2.3.6) is $L = \frac{1}{2}m\dot{x}^{\,2}e^{2\alpha x}$. To obtain the invariants of L we follow the procedure mentioned above.

Step 1: The invariance criterion (2.3.2) yields,

$$X^{[1]}L + LD_t\xi = D_t f(t,x)$$

$$\xi(t,x)\frac{\partial L}{\partial t} + \eta(t,x)\frac{\partial L}{\partial x} + \eta^{[1]}(t,x,\dot{x})\frac{\partial L}{\partial \dot{x}} + L\big(\xi_t + \dot{x}\xi_x\big) = f_t + \dot{x}f_x$$

$$0+\eta(t,x)(\alpha m\dot{x}^{\,2}e^{2\alpha x})+(\eta_t+\dot{x}\eta_x-\dot{x}\xi_t-\dot{x}^2\xi_x)(m\dot{x}e^{2\alpha x})+(\frac{1}{2}m\dot{x}^{\,2}e^{2\alpha x})(\xi_t+\dot{x}\xi_x) = f_t+\dot{x}f_x.$$

The above equation can be written as,

$$\dot{x}^{\,2}\big[\alpha m\eta(t,x)+m\eta_x-\frac{1}{2}m\xi_t\big]+\dot{x}\big[m\eta_t\big]+\dot{x}^3\big[-\frac{1}{2}m\xi_x\big] = e^{-2\alpha x}(f_t+\dot{x}f_x). \quad (2.3.7)$$

Step 2: Comparing the coefficients of Eq.(2.3.7) yields the following system of determining equations:

$$\alpha m\eta(t,x) + m\eta_x - \frac{1}{2}m\xi_t = 0, \quad (2.3.8)$$

$$m\eta_t = e^{-2\alpha x}f_x, \quad (2.3.9)$$

$$\xi_x = 0, \quad (2.3.10)$$

$$f_t = 0. \quad (2.3.11)$$

Step 3: Solving the above determining system yield the following result:

$$\xi(t,x) = c_1t^2 + c_2t + c_3,$$

$$\eta(t,x) = \big(c_4\frac{\alpha t}{m} + c_5\big)e^{-\alpha x} + \frac{tc_1}{\alpha} + \frac{c_2}{2\alpha},$$

$$f(t,x) = c_5 + c_4e^{\alpha x} + c_1\frac{m}{2\alpha^2}e^{2\alpha x}.$$

Step 4: We construct the symmetries as follows,

$$X_1 = t^2 \frac{\partial}{\partial t} + \frac{t}{\alpha} \frac{\partial}{\partial x} \ ,$$

$$X_2 = t \frac{\partial}{\partial t} + \frac{1}{2\alpha} \frac{\partial}{\partial x} \ ,$$

$$X_3 = \frac{\partial}{\partial t} \ ,$$

$$X_4 = \frac{\alpha t}{m} e^{-\alpha x} \frac{\partial}{\partial x} \ ,$$

$$X_5 = e^{-\alpha x} \frac{\partial}{\partial x} \ .$$

The gauge functions corresponding to the above symmetries are given respectively as,

$$f_1(t,x) = \frac{m}{2\alpha^2} e^{2\alpha x}, \quad f_2(t,x) = 0, \quad f_3(t,x) = 0, \quad f_4(t,x) = e^{\alpha x}, \quad f_5(t,x) = 0.$$

The list of symmetries both in differential and group form, along with their corresponding gauge functions, are shown in table 2.1 below:

Noether symmetry (defferential form)	Noether symmetry (group form)	gauge function
$X_1 = t^2 \frac{\partial}{\partial t} + \frac{t}{\alpha} \frac{\partial}{\partial x}$	$(\bar{t}, \bar{x}) = \left(\frac{t}{1-\varepsilon t}, \frac{t}{\alpha}\varepsilon + x\right)$	$f_1 = \frac{m}{2\alpha^2} e^{2\alpha x}$
$X_2 = t \frac{\partial}{\partial t} + \frac{1}{2\alpha} \frac{\partial}{\partial x}$	$(\bar{t}, \bar{x}) = \left(te^{\varepsilon}, \frac{\varepsilon}{2\alpha} + x\right)$	$f_2 = 0$
$X_3 = \frac{\partial}{\partial t}$	$(\bar{t}, \bar{x}) = \left(t + \varepsilon, x\right)$	$f_3 = 0$
$X_4 = \frac{\alpha t}{m} e^{-\alpha x} \frac{\partial}{\partial x}$	$(\bar{t}, \bar{x}) = \left(t, \frac{1}{\alpha} \ln\left[\frac{\alpha^2 t \varepsilon}{m} + e^{\alpha x}\right]\right)$	$f_4 = e^{\alpha x}$
$X_5 = e^{-\alpha x} \frac{\partial}{\partial x}$	$(\bar{t}, \bar{x}) = \left(t, \frac{1}{\alpha} \ln\left[\alpha\varepsilon + e^{\alpha x}\right]\right)$	$f_5 = 0$

Tab. 2.1: Noether symmetries of Eq.(2.3.6)

2.4 *Second Noether Theorem*

Theorem 2.4.1. Given a Lagrangian $L(t, x, \dot{x})$, a Noether symmetry,

$$X = \xi(t, x)\frac{\partial}{\partial t} + \eta(t, x)\frac{\partial}{\partial x} \ ,$$

and corresponding gauge function $f(t, x)$, then there exists a first integral given by,

$$I = L\xi + (\eta - \dot{x}\xi)\frac{\partial L}{\partial \dot{x}} - f(t, x). \tag{2.4.1}$$

Obviously, theorem 2.4.1 can be applied to construct conservation laws corresponding to a given Noether symmetry as we will see in the next example.

Example 2.4.1

We consider again Eq.(2.3.6) of example 2.3.1 with lagrangian $L = \frac{1}{2}m\dot{x}^2 e^{2\alpha x}$. We can construct the conservation law corresponding to the first derived symmetry,

$$X_1 = t^2\frac{\partial}{\partial t} + \frac{t}{\alpha}\frac{\partial}{\partial x} \ , \tag{2.4.2}$$

which corresponds to the gauge function, $f_1(t, x) = \frac{m}{2\alpha^2}e^{2\alpha x}$, by utilizing the formula of the first integral (2.4.1). Notice that for the symmetry X_1, $\xi = t^2$ and $\eta = \frac{t}{\alpha}$. Substituting in the formula (2.4.1), we obtain the following:

$$I = t^2\big(\frac{1}{2}m\dot{x}^2 e^{2\alpha x}\big) + m\dot{x}(\frac{t}{\alpha} - \dot{x}t^2)e^{2\alpha x} - \frac{m}{2\alpha^2}e^{2\alpha x}.$$

Therefore, the conserved form is derived as,

$$I = \big(\frac{mt}{\alpha}\dot{x} - \frac{m}{2\alpha^2} - \frac{1}{2}mt^2\dot{x}^2\big)e^{2\alpha x}. \tag{2.4.3}$$

One can easily verify that the quantity I, given by Eq.(2.4.3), does not change as time evolves by simply differentiating I with respect to t to get:

$$D_t I = D_t\big[\big(\frac{mt}{\alpha}\dot{x} - \frac{m}{2\alpha^2} - \frac{1}{2}mt^2\dot{x}^2\big)e^{2\alpha x}\big]$$

$$= (\frac{m}{\alpha}t\ddot{x} + mt\dot{x}^{\,2} - mt^2\dot{x}\ddot{x} - \alpha mt^2\dot{x}^{\,3})e^{2\alpha x}$$

$$= \frac{mt}{\alpha}e^{2\alpha x}(\ddot{x} + \alpha\dot{x}^{\,2}) - mt^2\dot{x}(\ddot{x} + \alpha\dot{x}^{\,2})e^{2\alpha x}$$

$$= \frac{mt}{\alpha}e^{2\alpha x}(0) - mt^2\dot{x}e^{2\alpha x}(0) = 0.$$

Therefore, the quantity I is conserved.

In this dissertation, we don't restrict our attention to Noether symmetries only, but we also deal with the well-known Lie symmetries. The Lie symmetries are extensively studied in many sources, see for example [12–22].

3. CLASSIFICATION OF VARIATIONAL CONSERVATION LAWS OF PLANE SYMMETRIC NON-STATIC SPACETIMES

The aim of this chapter is to give a complete classification of plane symmetric non-static spacetimes by the Noether symmetries admitted by them. In the next section, we give a quick review on the literature of the subject. In section 3.2, we briefly give the problem formulation. Derivation of determining equations for the Noether symmetries of the considered spacetime is given in section 3.3 and their solution is presented in section 3.4. The minimal and maximal symmetry algebra are derived in section 3.5. Application of the obtained results to specific interesting plane symmetric spacetimes is given in section 3.6. Finally, we conclude the chapter with a summary and discussion of the obtained results.

3.1 ***Introduction***

General Relativity is the most accurate theory of gravity in modern physics. Besides its accuracy, the theory has a reputation of being the most beautiful physical theory ever created. The beauty of the theory arises from its elegant description of gravitation in terms of geometry. Nevertheless, the elegance of the theory is just the tip of the iceberg which hides beneath it a deep mathematical structure that is hard to handle.

In fact, the mathematical structure of General Relativity is based on the Einstein field equations (EFEs) which is a system of ten non-linear PDEs describing gravitation as a result of the interaction between spacetime geometry and the distribution of matter and energy in its vicinity. Due to the nonlinearity of EFEs, obtaining exact solutions in closed analytic forms is too difficult to the extent that only few interesting solutions have been discovered since the invention of the theory one hundred years ago [41]. Therefore, various approaches have been developed to obtain exact solutions to the EFEs. The most popular approach is to exploit the symmetry properties of spacetimes.

In fact symmetry plays a very central role in increasing our understanding of the physical world. It served as a guiding principle in the discovery of the most fundamental theories of modern physics. One of the most profound result about the role of symmetry in nature was uncovered by Noether theorem. The reader is referred to chapter 2 for more details on Noether theorem.

In the context of GR, studying the already existing solutions of EFEs in terms of the symmetry properties they possess constitutes a significant part of GR research [42–44]. In this context, many attempts have been made to classify various types of spacetime metrics in terms of the symmetries they admit . In particular, the classification of plane symmetric spacetimes in terms of their symmetries have attracted the attention of many researchers. For example, Ali and Feroze performed a classification of plane symmetric static spacetimes in terms of their Noether symmetries [45]. Furthermore, the isometries of plane symmetric spacetimes have been investigated by Feroze, Qadir and Ziad [46]. Some researchers have foucused their attention on specific interesting plane symmetric spacetimes. For instance, the Minkwoski, the Taub and the anti-deSitter spacetimes have been studied by Shamir, Jhangeer and Bhatti where their killing and Noether symmetries have been derived [47].

3.2 ***Problem Formulation***

A plane symmetric spacetime is a Lorentzian manifold that possesses a stress energy tensor and admits $SO(2) \otimes R^2$ as the minimal isometry group in such a way that the group orbits are spacelike surfaces of constant curvature [46]. The group $SO(2) \otimes R^2$ is denoted by $\langle G_3 \rangle$ where $SO(2)$ is interpreted as rotation, while R^2 represents the translation along spatial directions x and y.

In this chapter, we intend to extend the works of Ferooze, Qadir and others by giving a complete classification of the plane symmetric non-static spacetimes by the Noether symmetries admitted by them. This classification is made by obtaining the general solution for the determining equations resulting from the invariance criterion for Noether symmetries. The solution is expressed in terms of unknown functions which are subject to a system of differential constraints that will be derived. Finally, the obtained solution will be applied to interesting plane symmetric spacetimes that admit Killing vector fields where in each case, both the Noether symmetries and the killing vector fields will be determined.

In particular, this research investigates the plane symmetric non-static spacetimes of the form,

$$ds^2 = e^{\nu(t,z)}dt^2 - e^{\lambda(t,z)}[dx^2 + dy^2] - e^{\mu(t,z)}dz^2, \tag{3.2.1}$$

where the objectives are :

1) Derive the Noether symmetries of the plane symmetric spacetimes in its most general form (the non-static form).

2) Obtain Noether conservation laws governing all forms of plane symmetric spacetimes.

3) Recover all the results obtained in literature about the considered spacetimes in terms of their Noether symmetries and Killing vector fields.

3.3 *Deriving the Determining Equations*

Since the general plane symmetric non-static spacetime is given by the metric (3.2.1), its corresponding Lagrangian L is given by,

$$L = e^{\nu(t,z)}\dot{t}^{\,2} - e^{\lambda(t,z)}[\dot{x}^{\,2} + \dot{y}^{\,2}] - e^{\mu(t,z)}\dot{z}^{\,2}. \tag{3.3.1}$$

A Noether symmetry of a spacetime metric in general is a transformation that leaves its Lagrangian invariant. In particular, if X is a generator of the form,

$$X = \xi\frac{\partial}{\partial s} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial z}\ , \tag{3.3.2}$$

then X is said to be a Noether symmetry for the metric (3.2.1) if it satisfies the following invariance criterion:

$$X^{[1]}L + L(D_s\xi) = D_s f(t,x,y,z), \tag{3.3.3}$$

where $X^{[1]}$ is the first prolongation of the symmetry generator (3.3.2) given by

$$X^{[1]} = X + \tau^{[1]}\frac{\partial}{\partial \dot{t}} + \eta^{[1]}\frac{\partial}{\partial \dot{x}} + \omega^{[1]}\frac{\partial}{\partial \dot{y}} + \phi^{[1]}\frac{\partial}{\partial \dot{z}}\ , \tag{3.3.4}$$

and f is the gauge function. Notice that, D is the differential operator defined as,

$$D = \frac{\partial}{\partial s} + \dot{t}\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\ ,$$

where the dot "." denotes differentiation with respect to s. Furthermore, the coefficients ξ, τ, η, ω and ϕ are functions of s, t, x, y and z. Thus, their corresponding coefficients $\tau^{[1]}, \eta^{[1]}, \omega^{[1]}$ and $\phi^{[1]}$, which are functions of $s, t, x, y, z, \dot{t}, \dot{x}, \dot{y}$

and $\dot{z}$, are given by the following formulas:

$$\tau^{[1]} = D_s\tau - \dot{t}D_s\xi,$$

$$\eta^{[1]} = D_s\eta - \dot{x}D_s\xi,$$

$$\omega^{[1]} = D_s\omega - \dot{y}D_s\xi,$$

$$\phi^{[1]} = D_s\phi - \dot{z}D_s\xi.$$

Applying the invariance criterion (3.3.3) to the Lagrangian (3.3.1) gives rise to the following system of differential equations:

$$f_s = 0, \tag{3.3.5}$$

$$\xi_t = \xi_x = \xi_y = \xi_z = 0, \tag{3.3.6}$$

$$\tau\nu_t + \phi\nu_z + 2\tau_t - \xi_s = 0, \tag{3.3.7}$$

$$\tau\lambda_t + \phi\lambda_z + 2\eta_x - \xi_s = 0, \tag{3.3.8}$$

$$\tau\lambda_t + \phi\lambda_z + 2\omega_y - \xi_s = 0, \tag{3.3.9}$$

$$\tau\mu_t + \phi\mu_z + 2\phi_z - \xi_s = 0, \tag{3.3.10}$$

$$2\tau_s e^{\nu(t,z)} = f_t, \tag{3.3.11}$$

$$\tau_x e^{\nu(t,z)} - \eta_t e^{\lambda(t,z)} = 0, \tag{3.3.12}$$

$$\tau_y e^{\nu(t,z)} - \omega_t e^{\lambda(t,z)} = 0, \tag{3.3.13}$$

$$\tau_z e^{\nu(t,z)} - \phi_t e^{\mu(t,z)} = 0, \tag{3.3.14}$$

$$-2\eta_s e^{\lambda(t,z)} = f_x, \tag{3.3.15}$$

$$\eta_y + \omega_x = 0, \tag{3.3.16}$$

$$\eta_z e^{\lambda(t,z)} + \phi_x e^{\mu(t,z)} = 0, \tag{3.3.17}$$

$$-2\omega_s e^{\lambda(t,z)} = f_y, \tag{3.3.18}$$

$$\omega_z e^{\lambda(t,z)} + \phi_y e^{\mu(t,z)} = 0, \tag{3.3.19}$$

$$-2\phi_s e^{\nu(t,z)} = f_z. \tag{3.3.20}$$

3.4 ***Solution of the Determining Equations***

Since solving the determining system requires very tedious calculations, we will only give a very short summary of the approach that we have used to solve the system. The key step that will be followed in this regard is to integrate equations (3.3.11), (3.3.15), (3.3.18) and (3.3.20) with respect to s respectively to obtain the following equations:

$$\tau = \frac{1}{2}e^{-\nu} f_t s + A(t,x,y,z), \tag{3.4.1}$$

$$\eta = -\frac{1}{2}e^{-\lambda} f_x s + B(t,x,y,z), \tag{3.4.2}$$

$$\omega = -\frac{1}{2}e^{-\lambda} f_y s + C(t,x,y,z), \tag{3.4.3}$$

$$\phi = -\frac{1}{2}e^{-\mu} f_z s + D(t,x,y,z). \tag{3.4.4}$$

Consequently, equations (3.4.1), (3.4.2), (3.4.3) and (3.4.4) imply the following:

$$\tau_{ss} = \eta_{ss} = \omega_{ss} = \phi_{ss} = 0. \tag{3.4.5}$$

Notice that Eq.(3.3.6) implies that ξ is a function of s only. Thus, we can determine the function ξ explicitly by differentiating Eq.(3.3.7) with respect to s twice to get $\xi_{sss} = 0$. Therefore, the function ξ is given by,

$$\xi = \frac{c_1}{2} s^2 + c_2 s + c_3. \tag{3.4.6}$$

It remains for us to determine the functions A, B, C and D arising in equations (3.4.1) to (3.4.4). This can be done by plugging these equations in the determining system and after lengthy calculations one arrives at the following results:

$$A(t,x,y,z) = \frac{e^{\lambda-\nu}}{2} g_t^{[1]}(t,z)(x^2+y^2) + e^{\lambda-\nu} y g_t^{[2]}(t,z) + e^{\lambda-\nu} x g_t^{[3]}(t,z) + \alpha(t,z), \tag{3.4.7}$$

$$B(t,x,y,z) = \frac{c_4}{2} x y^2 + \frac{c_5}{2} y^2 + c_6 x y + c_7 y + \frac{c_8}{6} x^3 + \frac{c_9}{2} x^2 + x g^{[1]}(t,z) + g^{[3]}(t,z), \tag{3.4.8}$$

$$C(t,x,y,z) = -\frac{c_4}{2} y x^2 - \frac{c_6}{2} x^2 - c_5 x y - c_7 x + \frac{c_4}{6} y^3 + \frac{c_6}{2} y^2 + y g^{[1]}(t,z) + g^{[2]}(t,z), \tag{3.4.9}$$

$$D(t,x,y,z) = -\frac{x^2+y^2}{2} e^{\lambda-\mu} g_z^{[1]}(t,z) - x e^{\lambda-\mu} g_z^{[3]}(t,z) - e^{\lambda-\mu} y g_z^{[2]}(t,z) + r(t,z). \tag{3.4.10}$$

Notice that $g^{[1]}(t,z), g^{[2]}(t,z), g^{[3]}(t,z), r(t,z)$ and $\alpha(t,z)$ are unknown functions. These functions can not be determined for the general metric (3.2.1) as they are subject to differential constrains that depend on the coefficients of the metric. Therefore, these functions can be determined only for explicit forms of the metric (3.2.1), i.e we need to substitute specific values for the functions ν, λ and μ.

Furthermore, our calculations have shown that the gauge function f has two different forms. Either f is a function of t and z only or it is a function of all the independent variables, i.e t, x, y and z. In particular to determine the function f, we need to define the following four quantities:

$$E_1 = \nu_z\lambda_t - 2\lambda_{tz} - \lambda_t\lambda_z + \mu_t\lambda_z, \tag{3.4.11}$$

$$E_2 = \nu_t\lambda_t - 2\lambda_{tt} + \nu_z\lambda_z e^{\nu-\mu} - \lambda_t^2, \tag{3.4.12}$$

$$E_3 = \lambda_t^2 e^{-\nu} - \lambda_z^2 e^{-\mu}, \tag{3.4.13}$$

$$E_4 = \mu_t\lambda_t e^{\mu-\nu} + \mu_z\lambda_z - 2\lambda_{zz} - \lambda_z^2. \tag{3.4.14}$$

Then, the gauge function f is classified according to the following two cases:

Case 1 : If $E_j = 0$ for $j = 1, 2, 3, 4$ then,

$$f = N(t,z) + e^{\frac{\lambda(t,z)}{2}}[\frac{c_8}{2}x^2 + c_9 x + \frac{c_8}{2}y^2 + c_{10}y + c_{11}]. \tag{3.4.15}$$

Case 2 : If $E_j \neq 0$ for any j then,

$$f = N(t,z). \tag{3.4.16}$$

The function $N(t,z)$ is unknown function as well and its determination, like other unknown functions, can be done by solving the system of differential constraints that will be given later. Finally, in order to write the general solution of the determining system, we substitute equations (3.4.7), (3.4.8), (3.4.9) and (3.4.10) in equations (3.4.1), (3.4.2), (3.4.3) and (3.4.4) respectively to obtain:

$$\begin{aligned}\tau = &\frac{e^{-\nu}}{2}N_t(t,z)s + \frac{e^{\lambda-\nu}}{2}(x^2+y^2)g_t^{[1]}(t,z) + e^{\lambda-\nu}yg_t^{[2]}(t,z)\\ &+xe^{\lambda-\nu}g_t^{[3]}(t,z) + \alpha(t,z) + \frac{\lambda_t}{4}e^{-\nu}s[f - N(t,z)],\end{aligned} \tag{3.4.17}$$

$$\eta = \frac{C_5}{2}y^2 + c_6xy + c_7y - \frac{c_5}{2}x^2 + xg^{[1]}(t,z) + g^{[3]}(t,z) - \frac{s}{2}e^{-\lambda(t,z)}f_x, \tag{3.4.18}$$

$$\omega = \frac{-C_6}{2}x^2 - c_5xy - c_7x + \frac{c_6}{2}y^2 + yg^{[1]}(t,z) + g^{[2]}(t,z) - \frac{s}{2}e^{-\lambda(t,z)}f_y, \quad (3.4.19)$$

$$\begin{aligned}\phi = &\frac{-e^{-\mu}}{2}N_z(t,z)s - \frac{e^{\lambda-\mu}}{2}(x^2+y^2)g_z^{[1]}(t,z) - e^{\lambda-\mu}xg_z^{[3]}(t,z) \\ &-e^{\lambda-\nu}yg_z^{[2]}(t,z) + r(t,z) - \frac{\lambda_z}{4}e^{-\mu}s[f - N(t,z)].\end{aligned} \quad (3.4.20)$$

Notice that the function ξ was already determined by Eq.(3.4.6).

The system of differential constraints can be derived by substituting equations (3.4.17) to (3.4.20) and Eq.(3.4.6) in the determining system to obtain the following system:

$$(\lambda_z - \nu_z)g_t^{[i]}(t,z) + (\lambda_t - \mu_t)g_z^{[i]}(t,z) + 2g_{tz}^{[i]}(t,z) = 0, \quad (3.4.21)$$

$$\mu_t e^{\mu-\nu}g_t^{[i]}(t,z) + \mu_z g_z^{[i]}(t,z) - 2\lambda_z g_z^{[i]}(t,z) - 2g_{zz}^{[i]}(t,z) = 0, \quad (3.4.22)$$

$$\lambda_t e^{-\nu}g_t^{[i]}(t,z) - \lambda_z e^{-\mu}g_z^{[i]}(t,z) + 2e^{-\lambda}k_i = 0, \quad (3.4.23)$$

$$\nu_z e^{\nu-\mu}g_z^{[i]}(t,z) + \nu_t g_t^{[i]}(t,z) - 2\lambda_t g_t^{[i]}(t,z) - 2g_{tt}^{[i]}(t,z) = 0, \quad (3.4.24)$$

$$\mu_t N_z(t,z) + \nu_z N_t(t,z) - 2N_{tz}(t,z) = 0, \quad (3.4.25)$$

$$\mu_t e^{-\nu}N_t(t,z) + \mu_z e^{-\mu}N_z(t,z) - 2e^{-\mu}N_{zz}(t,z) - 2c_1 = 0, \quad (3.4.26)$$

$$\nu_t e^{-\nu}N_t(t,z) + \nu_z e^{-\mu}N_z(t,z) - 2e^{-\nu}N_{tt}(t,z) + 2c_1 = 0, \quad (3.4.27)$$

$$\lambda_t e^{-\nu}N_t(t,z) - \lambda_z e^{-\mu}N_z(t,z) - 2e^{-\lambda}f_{xx}(t,x,y,z) - 2c_1 = 0, \quad (3.4.28)$$

$$e^{\nu}\alpha_z(t,z) = e^{\mu}r_t(t,z), \quad (3.4.29)$$

$$\mu_t\alpha(t,z) + \mu_z r(t,z) + 2r_z(t,z) - c_2 = 0, \quad (3.4.30)$$

$$\lambda_t\alpha(t,z) + \lambda_z r(t,z) + 2g^{[1]}(t,z) - c_2 = 0, \quad (3.4.31)$$

$$\nu_t\alpha(t,z) + \nu_z r(t,z) + 2\alpha_t(t,z) - c_2 = 0, \quad (3.4.32)$$

where $i = 1, 2, 3$ and $K_i s$ are constants given as follows, $k_1 = 0, k_2 = c_6$ and $k_3 = -c_5$.

In summary, we can obtain the Noether symmetries for any plane symmetric spacetime by following the six steps given below:

Step 1: Substitute the coefficients ν, λ and μ in equations (3.4.11) to (3.4.14) to determine the values of E_1 to E_4.

Step 2: Based on the values of $E_i s$, we determine the gauge function f according to the two cases mentioned earlier.

Step 3: Once f is determined, we substitute for f as well as the coefficients ν, λ and μ in the system of differential constraints given by equations (3.4.21) to (3.4.32).

Step 4: We solve the system of differential constraints to determine the unknown functions, namely, $g^{[i]}(t, z), N(t, z), \alpha(t, z)$ and $r(t, z)$.

Step 5: Once we determine the unknown functions, we plug them in the general solution given by equations (3.4.6) and (3.4.17) to (3.4.20).

Step 6: We construct the Noether symmetry corresponding to each constant in the derived solution of the determining system by substituting in the Noether symmetry generator (3.3.2).

3.5 Minimal and Maximal sets of Noether symmetries

The minimal set of Noether symmetries is the set containing the Noether symmetries admitted by arbitrary coefficients of the considered spacetimes given by metric (3.2.1). In fact, one can show that if the functions ν, λ and μ are arbitrary then the only possible solution of the system of differential constraints is given by $g^{[1]}(t, z) = \alpha(t, z) = r(t, z) = 0$ with the functions $g^{[2]}(t, z), g^{[3]}(t, z)$ and N(t,z) being constants where $c_1 = c_2 = c_5 = c_6 = 0$. Substituting the solution in equation (3.4.6) and equations (3.4.17) to (3.4.20) leads to the following result:

$$\xi = c_3, \qquad \tau = \phi = 0, \qquad \eta = c_7 y + c_8, \qquad \omega = -c_7 x + c_9.$$

From the above solution we conclude that the minimal set of Noether symmetries for the non-static plane symmetric spacetimes is generated by the group $\langle G_4 \rangle$ given by,

$$X_1 = \frac{\partial}{\partial s}\ , \qquad X_2 = \frac{\partial}{\partial x}\ , \qquad X_3 = \frac{\partial}{\partial y}\ , \qquad X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\ .$$

Notice that the symmetries X_2, X_3 and X_4 are Killing vector fields while X_1 is not a Killing vector, and all the elements of $\langle G_4 \rangle$ corresponds to a constant value of the gauge function.
The conservation laws for the minimal algebra $\langle G_4 \rangle$ is constructed using Noether theorem in the table 3.1 below.

No.	Noether symmetry	Conserved form
1	$\frac{\partial}{\partial s}$	$e^{\nu(t,z)}\dot{t}^{\,2} - e^{\lambda(t,z)}[\dot{x}^{\,2} + \dot{y}^{\,2}] - e^{\mu(t,z)}\dot{z}^{\,2}$
2	$\frac{\partial}{\partial x}$	$e^{\lambda(t,z)}\dot{x}$
3	$\frac{\partial}{\partial y}$	$e^{\lambda(t,z)}\dot{y}$
4	$y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$	$e^{\lambda(t,z)}(x\dot{y} - y\dot{x})$

Tab. 3.1: Conserved forms of the metric (3.2.1) corresponding to symmetries of $\langle G_4 \rangle$

The maximal set of Noether symmetries is obtained when the distortion of the spacetime geometry, created by curvature, is completely eliminated. That means that the zero curvature spacetime possesses the maximum number of Noether symmetries. This spacetime is given by the flat Minkowski spacetime given by,

$$ds^2 = dt^2 + dx^2 + dy^2 + dz^2. \tag{3.5.1}$$

The maximal set contains 17 symmetries, including the minimal group $\langle G_4 \rangle$ and 13 extra symmetries given as follows:

$$X_5 = \frac{\partial}{\partial t}, \qquad X_6 = s\frac{\partial}{\partial t}, \qquad X_7 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \qquad X_8 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x},$$

$$X_9 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \qquad X_{10} = s\frac{\partial}{\partial x}, \qquad X_{11} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad X_{12} = s\frac{\partial}{\partial y},$$

$$X_{13} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad X_{14} = s\frac{\partial}{\partial z}, \qquad X_{15} = \frac{\partial}{\partial z},$$

$$X_{16} = s^2\frac{\partial}{\partial s} + ts\frac{\partial}{\partial t} + xs\frac{\partial}{\partial x} + ys\frac{\partial}{\partial y} + zs\frac{\partial}{\partial z},$$

$$X_{17} = 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

Notice that the symmetries involving the arc length variable s are not killing vector fields while all the remaining symmetries are killing vectors which means that the Minkowski spacetime involves 10 killing vectors. The gauge function corresponding to the symmetries X_6, X_{10}, X_{14} and X_{16} are given respectively as follows, $f_6 = t, \qquad f_{10} = x, \qquad f_{14} = z, \qquad f_{16} = \frac{1}{2}t^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2.$ The remaining symmetries correspond to a constant value of the gauge function.

3.6 Noether and Killing symmetries for Plane Symmetric Spacetimes

In this section, we apply the results obtained in section 3.4 to perform a classification of Noether symmetries of the metric (3.2.1). In particular, we focuss our attention on the metrics admitting Killing vectors. Since the minimal algebra $\langle G_4 \rangle$ has been given in section 3.5, we will only mention in this section the extra (non-minimal) symmetries.

Case 1 Spacetimes admitting 11 Noether symmetries

a) The deSitter spacetime , given by the metric,

$$ds^2 = dt^2 - e^{\frac{2t}{a}}[dx^2 + dy^2 + dz^2] \qquad (a \neq 0), \tag{3.6.1}$$

admits 10 Killing vector fields and one non-Killing vector. The Killing vectors include the three Killing vectors of $\langle G_4 \rangle$ and 7 additional vectors given as follows:

$$X_5 = 2yz\frac{\partial}{\partial z} - 2ay\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x} + (y^2 - x^2 - z^2 + a^2 e^{-\frac{2}{a}t})\frac{\partial}{\partial y}, \quad X_6 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},$$

$$X_7 = \frac{\partial}{\partial z}, \quad X_8 = 2ax\frac{\partial}{\partial t} - 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} + (y^2 - x^2 + z^2 - a^2 e^{-\frac{2}{a}t})\frac{\partial}{\partial x},$$

$$X_9 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad X_{10} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - a\frac{\partial}{\partial t},$$

$$X_{11} = 2yz\frac{\partial}{\partial y} - 2az\frac{\partial}{\partial t} + 2xz\frac{\partial}{\partial x} + (z^2 - x^2 - y^2 + a^2 e^{-\frac{2}{a}t})\frac{\partial}{\partial z}.$$

Recall that the only non-Killing vector is given by $\frac{\partial}{\partial s}$ which belong to the group $\langle G_4 \rangle$. Notice that all the Noether symmetries mentioned above correspond to a constant value of the gauge function.

b) The anti-desitter spacetime given by,

$$ds^2 = e^{\frac{2z}{a}}[dt^2 - dx^2 - dy^2] - dz^2 \qquad (a \neq 0), \tag{3.6.2}$$

admits also 10 Killing vectors, where the the Killing symmetries other than minimal are:

$$X_5 = tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} - at\frac{\partial}{\partial z} + (x^2 + y^2 + t^2 + a^2 e^{-\frac{2}{a}z})\frac{\partial}{\partial t}, \quad X_6 = \frac{\partial}{\partial t},$$

$$X_7 = 2tx\frac{\partial}{\partial t} - 2ax\frac{\partial}{\partial z} + 2xy\frac{\partial}{\partial y} - (y^2 - x^2 - t^2 + a^2 e^{-\frac{2}{a}z})\frac{\partial}{\partial x},$$

$$X_8 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - a\frac{\partial}{\partial z},$$

$$X_9 = 2ty\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x} - 2ay\frac{\partial}{\partial z} - (x^2 - y^2 - t^2 + a^2 e^{-\frac{2}{a}z})\frac{\partial}{\partial y},$$

$$X_{10} = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}, \qquad X_{11} = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}.$$

Similarly, all the symmetries for the anti-desitter spacetime correspond to a constant value of the gauge function.

Case 2 Spacetimes admitting 9 Noether symmetries

a) The spacetime given by,

$$ds^2 = dt^2 - e^{\frac{2t}{a}}[dx^2 + dy^2] - dz^2 \qquad (a \neq 0), \tag{3.6.3}$$

admits nine Noether symmetries, where 7 of them are Killing. The non-minimal Killing are:

$$X_5 = -2ay\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x} + (y^2 - x^2 + a^2 e^{-\frac{2}{a}t})\frac{\partial}{\partial y}, \qquad X_6 = \frac{\partial}{\partial z},$$

$$X_7 = 2ax\frac{\partial}{\partial t} - xy\frac{\partial}{\partial y} + (y^2 - x^2 - a^2 e^{-\frac{2}{a}t})\frac{\partial}{\partial x}, \qquad X_8 = -a\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

The non-Killing Noether symmetries are two. The non-minimal one is $X_9 = s\frac{\partial}{\partial z}$. The gauge function corresponding to X_9 is $f_9 = -2z$, while the gauge function corresponding to the remaining symmetries is constant.

b) The anti-Einstein spacetime, given by,

$$ds^2 = dt^2 - e^{\frac{-2z}{a}}[dx^2 + dy^2] - dz^2 \qquad (a \neq 0), \tag{3.6.4}$$

admits 9 Noether symmetries as well. In particular, it possesses 7 Killing vectors corresponding to a constant gauge function, where the extra Killing are:

$$X_5 = 2xy\frac{\partial}{\partial y} + 2ax\frac{\partial}{\partial z} - (y^2 - x^2 + a^2 e^{\frac{2}{a}z})\frac{\partial}{\partial x}, \qquad X_6 = \frac{\partial}{\partial t},$$

$$X_7 = 2xy\frac{\partial}{\partial x} + 2ay\frac{\partial}{\partial z} + (y^2 - x^2 + a^2 e^{\frac{2}{a}z})\frac{\partial}{\partial y}, \qquad X_8 = a\frac{\partial}{\partial z} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

The remaining two symmetries are non-Killing, where the extra one is given by $X_9 = s\frac{\partial}{\partial t}$ with a gauge term given by $f_9 = 2t$.

c) The spacetime given by,

$$ds^2 = \cosh^2(\frac{z}{a})dt^2 - [dx^2 + dy^2 + dz^2] \qquad (a \neq 0), \tag{3.6.5}$$

admits 6 Killing vectors of constant gauge function, where the extra Killing vectors are:

$$X_5 = \tanh(\frac{z}{a})\cos(\frac{t}{a})\frac{\partial}{\partial t} + \sin(\frac{t}{a})\frac{\partial}{\partial z}, \qquad X_6 = \frac{\partial}{\partial t},$$

$$X_7 = -\tanh(\frac{z}{a})\sin(\frac{t}{a})\frac{\partial}{\partial t} + \cos\frac{t}{a}\frac{\partial}{\partial z}.$$

The extra non-Killing vectors are $X_8 = s\frac{\partial}{\partial x}$ which correspond to the gauge function $f_8 = -2x$ and $X_9 = s\frac{\partial}{\partial y}$ corresponding to $f_9 = -2y$.

d) The spacetime given by,

$$ds^2 = \cos^2(\frac{z}{a})dt^2 - [dx^2 + dy^2 + dz^2] \qquad (a \neq 0), \tag{3.6.6}$$

admits 6 Killing vectors of constant gauge function, where the extra Killing are:

$$X_5 = \tan(\frac{z}{a})\cosh(\frac{t}{a})\frac{\partial}{\partial t} + \sinh(\frac{t}{a})\frac{\partial}{\partial z}, \qquad X_6 = \frac{\partial}{\partial t},$$

$$X_7 = \tan(\frac{z}{a})\sinh(\frac{t}{a})\frac{\partial}{\partial t} + \cosh\frac{t}{a}\frac{\partial}{\partial z}.$$

The non-Killing Noether symmetries are the same as the symmetries X_8 and X_9 of the metric (3.6.5).

e) The spacetime given by,

$$ds^2 = e^{\frac{2z}{a}} dt^2 - [dx^2 + dy^2 + dz^2] \qquad (a \neq 0), \tag{3.6.7}$$

admits 6 Killing vectors of constant gauge function, where the non-minimal Killing vectors are:

$$X_5 = (ae^{-\frac{2z}{a}} + \frac{t^2}{a})\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \qquad X_6 = \frac{t}{a}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}, \qquad X_7 = \frac{\partial}{\partial t}.$$

The non-Killing symmetries are also the same as X_8 and X_9 of the metric (3.6.5).

f) The spacetime given by,

$$ds^2 = dt^2 - [dx^2 + dy^2] - \cosh^2(\frac{t}{a})dz^2 \qquad (a \neq 0), \tag{3.6.8}$$

admits 6 Killing symmetries of constant gauge function, where the extra Killing symmetries are given by:

$$X_5 = \tanh(\frac{t}{a})\cos(\frac{z}{a})\frac{\partial}{\partial z} + \sin(\frac{z}{a})\frac{\partial}{\partial t}, \qquad X_6 = \frac{\partial}{\partial z},$$

$$X_7 = \tanh(\frac{t}{a})\sin(\frac{z}{a})\frac{\partial}{\partial z} - \cos\frac{z}{a}\frac{\partial}{\partial t}.$$

The non-Killing symmetries are also similar to the symmetries of the metric (3.6.5).

g) The spacetime given by,

$$ds^2 = dt^2 - [dx^2 + dy^2] - \cos^2(\frac{t}{a})dz^2 \qquad (a \neq 0), \tag{3.6.9}$$

possesses 6 Killing symmetries of constant gauge term, where the non-minimal

Killing symmetries are given by:

$$X_5 = a\sinh(\frac{z}{a})\frac{\partial}{\partial t} + a\tan(\frac{t}{a})\cosh(\frac{z}{a})\frac{\partial}{\partial z}, \qquad X_6 = \frac{\partial}{\partial z},$$

$$X_7 = a\cosh(\frac{z}{a})\frac{\partial}{\partial t} + a\tan\frac{t}{a}\sinh(\frac{z}{a})\frac{\partial}{\partial z}.$$

The non-Killing symmetries are also similar to the symmetries of the metric (3.6.5).

h) The spacetime given by,

$$ds^2 = dt^2 - [dx^2 + dy^2] - e^{\frac{2t}{a}}dz^2 \qquad (a \neq 0), \qquad (3.6.10)$$

admits 6 Killing vectors of constant gauge term, where the non-minimal Killing vectors are given by:

$$X_5 = (\frac{z^2}{a} - ae^{\frac{-2t}{a}})\frac{\partial}{\partial z} + 2z\frac{\partial}{\partial t}, \qquad X_6 = z\frac{\partial}{\partial z} - a\frac{\partial}{\partial t}, \qquad X_7 = \frac{\partial}{\partial z}.$$

The non-Killing symmetries are also similar to the symmetries of the metric (3.6.5).

Case 3 Spacetimes admitting 7 Noether symmetries

a) The spacetime given by,

$$ds^2 = e^{2g(z)}dt^2 - e^{2g(z)+\frac{2t}{a}}[dx^2 + dy^2] - dz^2 \qquad (a \neq 0), \qquad (3.6.11)$$

admits 7 Noether symmetries. All the non-minimal symmetries are Killing vectors of constant gauge function, given by:

$$X_5 = (y^2 - x^2 + a^2e^{\frac{-2t}{a}})\frac{\partial}{\partial y} - 2ay\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x}, \qquad X_6 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - a\frac{\partial}{\partial t},$$

$$X_7 = (y^2 - x^2 - a^2 e^{\frac{-2t}{a}})\frac{\partial}{\partial x} + 2ax\frac{\partial}{\partial t} - 2xy\frac{\partial}{\partial y}.$$

b) The spacetime given by,

$$ds^2 = dt^2 - e^{2g(t)+\frac{2z}{a}}[dx^2 + dy^2] - e^{2g(t)}dz^2 \qquad (a \neq 0), \qquad (3.6.12)$$

admits 7 Noether symmetries, where all the non-minimal symmetries are Killing vectors of constant gauge function, given by:

$$X_5 = (y^2 - x^2 + a^2 e^{\frac{-2z}{a}})\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y} + 2ax\frac{\partial}{\partial z}, \qquad X_6 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - a\frac{\partial}{\partial z},$$

$$X_7 = (y^2 - x^2 - a^2 e^{\frac{-2z}{a}})\frac{\partial}{\partial y} + 2xy\frac{\partial}{\partial x} - 2ay\frac{\partial}{\partial z}.$$

c) The spacetime given by,

$$ds^2 = e^{2r(t,z)}dt^2 - e^{\frac{-2}{a}t}[dx^2 + dy^2] - e^{2p(t-z)}dz^2 \qquad (a \neq 0), \qquad (3.6.13)$$

subject to the following condition:

$$e^{2p(t,z)+\frac{2t}{a}} = e^{2r(t,z)+\frac{2t}{a}} - \frac{2}{a}\int e^{2r(t,z)+\frac{2t}{a}}dt,$$

admits 7 Noether symmetries, where all the non-minimal symmetries are Killing vectors of constant gauge function, given by:

$$X_5 = (y^2 - x^2 + a^2(e^{2r(t,z)+\frac{2t}{a}} - e^{2p(t,z)+\frac{2t}{a}}))\frac{\partial}{\partial y} + 2ay\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x} - 2ay\frac{\partial}{\partial z},$$

$$X_6 = a\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - a\frac{\partial}{\partial z},$$

$$X_7 = (y^2 - x^2 - a^2(e^{2r(t,z)+\frac{2t}{a}} - e^{2p(t,z)+\frac{2t}{a}}))\frac{\partial}{\partial x} - 2ax\frac{\partial}{\partial t} - 2xy\frac{\partial}{\partial y} + 2ax\frac{\partial}{\partial z}.$$

Case 4 Spacetimes admitting 6 Noether symmetries

a) The spacetime given by,

$$ds^2 = e^{\frac{2z}{a}} dt^2 - e^{\frac{2z}{b}} [dx^2 + dy^2] - dz^2 \qquad (a \neq 0, \quad b \neq 0, \quad a \neq b), \quad (3.6.14)$$

admits 6 Noether symmetries including two non-minimal symmetries, being Killing vectors of constant gauge term, and given by:

$$X_5 = \frac{b}{a} t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - b \frac{\partial}{\partial z}, \qquad X_6 = \frac{\partial}{\partial t}.$$

b) The spacetime given by,

$$ds^2 = dt^2 - e^{\frac{2t}{b}} [dx^2 + dy^2] - e^{\frac{2t}{a}} dz^2 \qquad (a \neq 0, \quad b \neq 0), \qquad (3.6.15)$$

admits 6 Noether symmetries including two non-minimal symmetries, being Killing vectors of constant gauge term, and given by:

$$X_5 = -b \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{b}{a} z \frac{\partial}{\partial z}, \qquad X_6 = \frac{\partial}{\partial z}.$$

Case 5 Spacetimes admitting 5 Noether symmetries

a) The spacetime given by,

$$ds^2 = e^{2v(z)} dt^2 - e^{2\lambda(z)} [dx^2 + dy^2] - dz^2, \qquad (3.6.16)$$

admits 5 Noether symmetries including just one non-minimal Killing symmetry of constant gauge function, given by,

$$X_5 = \frac{\partial}{\partial t}.$$

b) The spacetime given by,

$$ds^2 = dt^2 - e^{2\lambda(t)}[dx^2 + dy^2] - e^{2v(t)}dz^2, \qquad (3.6.17)$$

admits 5 Noether symmetries including just one non-minimal Killing symmetry of constant gauge function, given by:

$$X_4 = \frac{\partial}{\partial z}.$$

3.7 ***Conclusion***

In this chapter, we have obtained the general solution of the Noether symmetry equation for plane symmetric non-static spacetimes. This solution is expressed in terms of unknown functions that can be determined by solving the system of differential constraints provided in section 3.4. It has been shown that the number of Noether symmetries for the studied spacetimes in general ranges between 4 and 17. The minimal algebra, admitted by arbitrary coefficients of the metric (3.2.1), is generated by the group $\langle G_4 \rangle$, while the maximal algebra, spanned by 17 Noether symmetries, is admitted by Minkowski spacetime.

In fact, since the plane symmetric static spacetime is a special case of our considered spacetime metric (3.2.1) , one can recover all the results obtained in [45] and [47] by following the six steps for obtaining Noether symmetries we have provided in section 3.4. Furthermore, in section 3.6, we have recovered all the results obtained in [46], which provided the Killing vectors for plane symmetric non-static spacetimes, and we were able to extend these results by deriving the extra Noether symmetries that are non-killing vectors.

Consequently, this work represents the most general attempt to investigate the

Noether symmetries, the isometries and their corresponding variational conservation laws for plane symmetric non-static spacetimes.

4. CLASSIFICATION OF INVARIANCES OF SOME NONLINEAR FORCED AND DAMPED WAVE EQUATIONS

In this chapter, we address certain class of one dimensional damped wave equations. After giving a brief introduction of the subject in section 4.1, we formulate our problem in section 4.2. Then, the determining equations for the considered damped wave equation is derived in section 4.3. Section 4.4 is the main part of the chapter where we perform a Lie symmetry classification of the considered equation by solving its determining system. Some reductions and solutions of the damped wave equation are given in section 4.5. We conclude the chapter in section 4.6, where we present a brief summary of the results.

4.1 ***Introduction***

The damped wave equations are extremely important in modeling many interesting real life problems that arise in different areas of science and engineering. For example, damped wave equations can be used to predict neural activity using brain anatomy. Furthermore, They can be used to study wave behavior in quantum mechanical systems. One is led to such equations when studying some fundamental systems arising in atomic physics and electromagnetism or when studying heat conduction in relativistic systems.

Recall that the damped wave equations are used to model propagation of waves with friction or damping. In particular, when a physical system losses energy then the classical wave model is modified by adding a term representing material damping of the system. The solutions of the damped wave equations exhibit diffusion phenomena in that they behave like solutions of the corresponding heat equation.

Our aim in this chapter is to perform symmetry classification of certain class of damped wave equations. Recall that the symmetry classification of wave equations in general has been extensively investigated. In this context, Ibragimov gave a classification of different classes of wave and damped wave equations in terms of Lie point symmetries admitted by them [16–18]. In particular, he studied damped wave equations involving damping terms in different dimensions [18].

However, most of the attention in literature has been focused on the classification of classical wave equations. Thus, classification of the damped wave equations need to be paid more attention in order to understand the effect of coupling a damping term on the symmetry properties of wave equations.

4.2 ***Problem Formulation***

In this chapter, we address the problem of solving a one dimensional non-linear damped wave equation by considering damping and forcing terms as being functions of u. In particular, the equation,

$$u_{tt} + f(u)u_t = u_{xx} + g(u), \tag{4.2.1}$$

is investigated. In this contest, we provide a set of solutions of this equation by classifying both the damping coefficient $f(u)$ and the forcing terms $g(u)$. We also extend our investigations by performing different similarity reductions where exact solutions are obtained whereas possible .

4.3 *Deriving the Determining Equations*

The symmetry of Eq.(4.2.1) can be expressed via an infinitesimal symmetry generator associated with (x, t, u) space ,

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}. \tag{4.3.1}$$

Since Eq.(4.2.1) is a second order PDE, the symmetry generator (4.3.1) needs to be prolonged so that it involves the transformation of first and second order derivatives of u. This prolonged generator is given by,

$$X^{[2]} = X + \phi^t\frac{\partial}{\partial u_t} + \phi^x\frac{\partial}{\partial u_x} + \phi^{tt}\frac{\partial}{\partial u_{tt}} + \phi^{tx}\frac{\partial}{\partial u_{tx}} + \phi^{xx}\frac{\partial}{\partial u_{xx}}.$$

The coefficients ϕ^t, ϕ^x, ϕ^{tt}, ϕ^{tx}, and ϕ^{xx} can be obtained by the following formulae:

$$\phi^i = D_i(\phi - \xi\ u_x - \tau\ u_t) + \xi\ u_{x,i} + \tau\ u_{t,i}\ , \tag{4.3.2}$$

$$\phi^{ij} = D_i D_j(\phi - \xi\ u_x - \tau\ u_t) + \xi\ u_{x,ij} + \tau\ u_{t,ij}\ . \tag{4.3.3}$$

The invariance criterion for obtaining symmetries of Eq.(4.2.1) requires that the prolonged generator(vector field) vanishes when applied to the equation $F = u_{tt} + f(u)u_t - u_{xx} - g(u)$. This criterion leads to a system of eight pde's given as follows:

$$\tau_u = 0 = \xi_u = \phi_{uu}, \tag{4.3.4}$$

$$\xi_t - \tau_x = 0, \tag{4.3.5}$$

$$\tau_t - \xi_x = 0, \tag{4.3.6}$$

$$f_u\phi + f(u)\tau_t - \tau_{tt} + \tau_{xx} + 2\phi_{tu} = 0, \tag{4.3.7}$$

$$-f(u)\xi_t - \xi_{tt} + \xi_{xx} - 2\phi_{xu} = 0, \tag{4.3.8}$$

$$-g_u\phi \ - 2g(u)\tau_t \ + f(u)\phi_t \ + g(u)\phi_u \ + \phi_{tt} - \phi_{xx} = 0. \qquad (4.3.9)$$

4.4 *Classification of Symmetries of Eq.(4.2.1)*

In order to carry out classification, we need to solve the above over-determined system(Eq.(4.3.4) to Eq.(4.3.9)). This system can be solved by iteratively using the process of elimination. For this purpose, we first differentiate Eq.(4.3.7) with respect to u to get,

$$f_{uu}\phi \ + f_u\phi_u \ + f_u\tau_t \ = 0. \qquad (4.4.1)$$

We start solving by considering Eq.(4.4.1). We have two cases,viz., $f_u \neq 0$ or $f_u = 0$. We discuss each case in details as follows.

Case I $(f_u \neq 0)$

In this case, Eq.(4.4.1) can be re-cast in the form,

$$\frac{f_{uu}}{f_u}\phi \ = -\phi_u \ - \tau_t. \qquad (4.4.2)$$

From Eq.(4.4.2),'two' possibilities arise, namely, $\phi \ = 0$ or $\phi \ \neq 0$.

Case I-A $(\phi \ = 0)$

Substituting $\phi \ = 0$, in the system of determining equations, immediately implies that $\tau = c_1$ and $\xi = c_2$. We notice that, this solution is valid for arbitrary functions $f(u)$ and $g(u)$. Hence, this case provides the minimal set of symmetries generating the minimal group $< G2 >$ spanned by,

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}.$$

Case I-B $(\phi \neq 0)$

At this stage we re-cast Eq.(4.4.2) by ϕ to obtain,

$$\frac{f_{uu}}{f_u} = -\frac{\phi_u \ + \tau_t}{\phi}. \tag{4.4.3}$$

Since the left hand side of Eq.(4.4.3) depends only on u, it can be put in the form,

$$\frac{f_{uu}}{f_u} = \frac{-\phi_u \ - \tau_t}{\phi} = h(u). \tag{4.4.4}$$

Now, we consider the possible cases generated by Eq.(4.4.4). These cases are : $h(u) = 0$, $h(u) = constant \neq 0$ and $h(u)$ depends explicitly on u.

Case I-B-1. $(h(u) = 0)$

Then, Eq.(4.4.4) becomes,

$$\frac{f_{uu}}{f_u} = \frac{-\phi_u \ - \tau_t}{\phi} = 0. \tag{4.4.5}$$

Integrating Eq.(4.4.5) with respect to u and solving for f we obtain $f(u) = \alpha\, u+\beta$, where $u > -\frac{\beta}{\alpha}$. Similarly, if we solve for ϕ, we obtain $\phi = -\tau_t u + A(t, x)$. Substituting the expressions of f and ϕ in the determining system and after more manipulation, one finds that the solution for the determining system can be written as,

$$\tau = c_1 + c_3 t, \qquad \xi = c_2 + c_3 x, \qquad \phi = -c_3(u + \frac{\beta}{\alpha}).$$

Furthermore, the function g in this case is determined to be $g(u) = \gamma\, (\alpha\, u+\beta\,)^3$. Hence, the symmetry algebra for the case I-B-1 is generated by a group $< G3 >$,

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - (u + \frac{\beta}{\alpha})\frac{\partial}{\partial u}.$$

Case I-B-2 ($h(u) = \alpha \neq 0$)

In this case, Eq.(4.4.4) takes the following form:

$$\frac{f_{uu}}{f_u} = \frac{-\phi_u \;\; - \tau_t}{\phi} = \alpha. \tag{4.4.6}$$

Integrating Eq.(4.4.6) with respect to u and solving for f yields,

$$f(u) = \frac{\beta}{\alpha} e^{\alpha u} + \gamma, \tag{4.4.7}$$

such that $\frac{\beta}{\alpha} e^{\alpha u} > -\gamma$. Similarly, solving for ϕ gives $\phi_u = 0$, or equivalently, $\phi = A(t, x)$. Then, following the same procedure as in case I-B-1 leads to the following solution:

$$\tau = c_1 + \alpha c_3 t, \qquad \xi = c_2 + \alpha c_3 x, \qquad \phi = -c_3.$$

Consequently, Eq.(4.4.7) is reduced into $f(u) = \frac{\beta}{\alpha} e^{\alpha u}$, where $\frac{\beta}{\alpha} > 0$. Furthermore, the function g in this case is derived to be $g(u) = \sigma \ e^{2\alpha u}$. Therefore, the corresponding symmetries are given by,

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \alpha t \frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x} - \frac{\partial}{\partial u}.$$

Case I-B-3 ($h'(u) \neq 0$)

Differentiating Eq.(4.4.4) with respect to u, we obtain $\phi_u h(u) + \phi h'(u) = 0$, or equivalently,

$$\frac{\phi_u}{\phi} = -\frac{h'(u)}{h(u)}. \tag{4.4.8}$$

Integrating Eq.(4.4.8) with respect to u yields,

$$\phi = \frac{A(t, x)}{h(u)}. \tag{4.4.9}$$

In order to determine the function $h(u)$, we use Eqs.(4.4.9) and (4.3.4) to obtain,

$$h(u) = \frac{1}{\alpha u + \beta}. \tag{4.4.10}$$

Consequently, Eq.(4.4.9) becomes,

$$\phi = A(t, x)(\alpha u + \beta). \tag{4.4.11}$$

Furthermore, Eq.(4.4.4) can be written as,

$$\frac{f_{uu}}{f_u} = \frac{-\phi_u - \tau_t}{\phi} = \frac{1}{\alpha u + \beta}. \tag{4.4.12}$$

Integrating Eq.(4.4.12) with respect to u yields $f'(u) = \gamma(\alpha u + \beta)^{\frac{1}{\alpha}}$. Since the case $\alpha = -1$ does not provide explicit solution to the determining system, we restrict our attention to the case $\alpha \neq -1$ only. In this case, we obtain $f(u) = \frac{\gamma}{\alpha+1}(\alpha u + \beta)^{\frac{1}{\alpha}+1}$, such that the following restrictions are met: $\alpha \neq 0$, $\frac{\gamma}{\alpha+1} > 0$ and $u > -\frac{\beta}{\alpha}$. After some more manipulations, the functions τ, ξ, and ϕ are obtained as follows,

$$\tau = c_1 + (\alpha + 1)c_3 t, \qquad \xi = c_2 + (\alpha + 1)c_3 x, \qquad \phi = -c_3(\alpha u + \beta).$$

Then, Eq.(4.3.9) can be used to determine the function g. In particular, simple calculations show that $g(u) = \sigma(\alpha u + \beta)^{3+\frac{2}{\alpha}}$. Therefore, the symmetry algebra for this subcase is given by,

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = (\alpha + 1)t\frac{\partial}{\partial t} + (\alpha + 1)x\frac{\partial}{\partial x} - (\alpha u + \beta)\frac{\partial}{\partial u}.$$

Case II. $(f'(u) = 0)$

Let $f(u) = \alpha > 0$. Then, differentiating Eq.(4.3.9) with respect to u and after some manipulation, the obtained equation can be reduced to,

$$-g_{uu}\phi - 2\tau_t g_u + \alpha\phi_{tu} = 0. \tag{4.4.13}$$

Differentiating Eq.(4.4.13) with respect to u gives,

$$-g_{uuu}\phi - g_{uu}\phi_u - 2\tau_t g_{uu} = 0. \tag{4.4.14}$$

Notice that if $\phi = 0$ then we obtain again the principal algebra so we assume that $\phi \neq 0$. In this case, the Eq.(4.4.14) can be written as,

$$-g_{uuu} = \frac{\phi_u + 2\tau_t}{\phi} g_{uu}. \tag{4.4.15}$$

At this stage, we consider two possibilities: $g_{uu} = 0$ and $g_{uu} \neq 0$.

Case II-A. ($g_{uu} = 0$)

Then, the function g is given by $g(u) = \beta u + \rho$. Now, substituting for g in the determining system and making tedious calculations lead to the equation,

$$(\alpha^2 + 4\beta)\phi_{tu} = 0. \tag{4.4.16}$$

If $\beta \neq -\frac{\alpha^2}{4}$ then $\phi_{tu} = 0$, and consequently, the solution of the system in this case leads again to the minimal algebra obtained in case 1-A. Hence, we assume that $\beta = -\frac{\alpha^2}{4}$ which implies that $g(u) = -\frac{\alpha^2}{4}u + \rho$. Then, carrying out further calculations gives rise to the following solution:

$$\tau = c_1 + \frac{c_3}{\alpha}e^{-\frac{\alpha}{4}(t-x)} + \frac{c_4}{\alpha}e^{-\frac{\alpha}{4}(t+x)}, \qquad \xi = c_2 - \frac{c_3}{\alpha}e^{-\frac{\alpha}{4}(t-x)} + \frac{c_4}{\alpha}e^{-\frac{\alpha}{4}(t+x)},$$

$$\phi = (c_3 e^{-\frac{\alpha}{4}(t-x)} + c_4 e^{-\frac{\alpha}{4}(t+x)})(\beta - \frac{1}{2}u).$$

Thus, the derived solution leads to four-dimensional symmetry algebra generating the group $< G4 >$, given by,

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x},$$

$$X_3 = \frac{e^{-\frac{\alpha}{4}(t-x)}}{\alpha}\frac{\partial}{\partial t} - \frac{e^{-\frac{\alpha}{4}(t-x)}}{\alpha}\frac{\partial}{\partial x} + e^{-\frac{\alpha}{4}(t-x)}(\beta - \frac{1}{2}u)\frac{\partial}{\partial u},$$
$$X_4 = \frac{e^{-\frac{\alpha}{4}(t+x)}}{\alpha}\frac{\partial}{\partial t} + \frac{e^{-\frac{\alpha}{4}(t+x)}}{\alpha}\frac{\partial}{\partial x} + e^{-\frac{\alpha}{4}(t+x)}(\beta - \frac{1}{2}u)\frac{\partial}{\partial u}.$$

Case II-B. ($g_{uu} \neq 0$)

Then, Eq.(4.4.15) can be re-cast in the form

$$\frac{g_{uuu}}{g_{uu}} = -\frac{\phi_u + 2\tau_t}{\phi}. \tag{4.4.17}$$

Again since the left hand side of Eq.(4.4.17) depends only on u we can re-write the equation as,

$$\frac{g_{uuu}}{g_{uu}} = -\frac{\phi_u + 2\tau_t}{\phi} = h(u). \tag{4.4.18}$$

At this stage, three cases can be considered; $h(u) = 0$, $h(u) =$constant$\neq 0$ and $h'(u) \neq 0$

Case II-B-1 ($h(u) = 0$)

In this case, Eq.(4.4.18) takes the form,

$$\frac{g_{uuu}}{g_{uu}} = -\frac{\phi_u + 2\tau_t}{\phi} = 0. \tag{4.4.19}$$

Solving Eq.(4.4.19) for g gives $g(u) = \frac{\beta}{2}u^2 + \gamma u + \sigma$. Then, following the same procedure as in the previous cases provides the following solution:

$$\tau = c_1 + \frac{4}{\alpha}c_3 e^{\frac{\alpha}{4}(t+x)} + \frac{4}{\alpha}c_4 e^{\frac{\alpha}{4}(t-x)}, \qquad \xi = c_2 + \frac{4}{\alpha}c_3 e^{\frac{\alpha}{4}(t+x)} - \frac{4}{\alpha}c_4 e^{\frac{\alpha}{4}(t-x)},$$

$$\phi = (-2u - \frac{2\gamma}{\beta} - \frac{\alpha^2}{2\beta})(c_3 e^{\frac{\alpha}{4}(t+x)} + c_4 e^{\frac{\alpha}{4}(t-x)}).$$

Consequently, the obtained solution imposes certain restriction on the value of σ. In particular, σ must equal $\frac{\gamma^2}{2\beta} - \frac{\alpha^4}{32\beta}$, or equivalently, $g(u) = \frac{\beta}{2}u^2 + \gamma u + \frac{\gamma^2}{2\beta} - \frac{\alpha^4}{32\beta}$. At this moment we construct the symmetry algebra corresponding to the above solution as follows,

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x},$$

$$X_3 = \frac{4}{\alpha} e^{\frac{\alpha}{4}(t+x)} \frac{\partial}{\partial t} + \frac{4}{\alpha} e^{\frac{\alpha}{4}(t+x)} \frac{\partial}{\partial x} - (2u + \frac{2\gamma}{\beta} + \frac{\alpha^2}{2\beta}) e^{\frac{\alpha}{4}(t+x)} \frac{\partial}{\partial u},$$

$$X_4 = \frac{4}{\alpha} e^{\frac{\alpha}{4}(t-x)} \frac{\partial}{\partial t} - \frac{4}{\alpha} e^{\frac{\alpha}{4}(t-x)} \frac{\partial}{\partial x} - (2u + \frac{2\gamma}{\beta} + \frac{\alpha^2}{2\beta}) e^{\frac{\alpha}{4}(t-x)} \frac{\partial}{\partial u}.$$

Case II-B-2 $(h(u) = \beta \neq 0)$

In this case, Eq.(4.4.18) becomes,

$$\frac{g_{uuu}}{g_{uu}} = -\frac{\phi_u + 2\tau_t}{\phi} = \beta. \tag{4.4.20}$$

Solving Eq.(4.4.20) and substituting the solution in the determining system leads to the minimal symmetry algebra obtained in case I-A.

Case II-B-3. $(h'(u) \neq 0)$

Then, Differentiating Eq.(4.4.18) with respect to u and following the same procedure as in case I-B-3 lead to the following results: $\phi = \frac{A(t,x)}{h(u)}$ and $h(u) = \frac{1}{\sigma u + \beta}$. Consequently, Eq.(4.4.18) can be written as,

$$\frac{g_{uuu}}{g_{uu}} = \frac{1}{\sigma u + \beta}. \tag{4.4.21}$$

At this stage, one can show that if $\sigma = -1$ or $\sigma = -\frac{1}{2}$ then the condition $g_{uu} \neq 0$ is violated. Hence, we assume that $\sigma \notin \{-1, -\frac{1}{2}\}$. Then, solving Eq.(4.4.21) gives,

$$g(u) = \frac{\gamma}{(\sigma+1)(2\sigma+1)}(\sigma u+\beta)^{2+\frac{1}{\sigma}} + \nu u + \omega. \tag{4.4.22}$$

After carrying out tedious calculations, one arrives at the following solution:

$$\tau = c_1 - \frac{2\sigma}{\alpha} c_3 e^{\frac{\alpha}{4\sigma}(1+\sigma)(t+x)} - \frac{2\sigma}{\alpha} c_4 e^{\frac{\alpha}{4\sigma}(1+\sigma)(t-x)},$$

$$\xi = c_2 - \frac{2\sigma}{\alpha} c_3 e^{\frac{\alpha}{4\sigma}(1+\sigma)(t+x)} + \frac{2\sigma}{\alpha} c_4 e^{\frac{\alpha}{4\sigma}(1+\sigma)(t-x)},$$

$$\phi = (c_3 e^{\frac{\alpha}{4\sigma}(1+\sigma)(t+x)} + c_4 e^{\frac{\alpha}{4\sigma}(1+\sigma)(t-x)})(\sigma u+\beta).$$

Consequently, the above solution imposes certain restrictions on the values ν and ω of Eq.(4.4.22). In particular, Eq.(4.4.22) can be written as,

$$g(u) = \frac{\gamma}{(\sigma+1)(2\sigma+1)}(\sigma u+\beta)^{2+\frac{1}{\sigma}} - \frac{\alpha^2}{4} u - \frac{\alpha^2\beta}{4\sigma}. \tag{4.4.23}$$

Therefore, the corresponding symmetry algebra is given by,

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x},$$

$$X_3 = -\frac{2\sigma}{\alpha} e^{\frac{\alpha}{4\sigma}(1+\sigma)(t+x)} \frac{\partial}{\partial t} - \frac{2\sigma}{\alpha} e^{\frac{\alpha}{4\sigma}(1+\sigma)(t+x)} \frac{\partial}{\partial x} + e^{\frac{\alpha}{4\sigma}(1+\sigma)(t+x)}(\sigma u+\beta)\frac{\partial}{\partial u},$$

$$X_4 = -\frac{2\sigma}{\alpha} e^{\frac{\alpha}{4\sigma}(1+\sigma)(t-x)} \frac{\partial}{\partial t} + \frac{2\sigma}{\alpha} e^{\frac{\alpha}{4\sigma}(1+\sigma)(t-x)} \frac{\partial}{\partial x} + e^{\frac{\alpha}{4\sigma}(1+\sigma)(t-x)}(\sigma u+\beta)\frac{\partial}{\partial u}.$$

At this stage, we conclude section 4.4, where it is shown that the studied damped wave equation has at least two dimensional symmetry algebra generated by the group $< G2 >$. The cases that have been discussed in this section are presented in table 4.1, where for each case we presented the non-minimal symmetries only.

$f(u)$	$g(u)$	Non-minimal symmetries
arbitrary	arbitrary	No non-minimal symmetry
$\alpha u+\beta$	$\gamma(\alpha u+\beta)^3$	$X_3=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-(u+\frac{\beta}{\alpha})\frac{\partial}{\partial u}$
$\frac{\beta}{\alpha}e^{\alpha u}$	$\sigma e^{2\alpha u}$	$X_3=\alpha t\frac{\partial}{\partial t}+\alpha x\frac{\partial}{\partial x}-\frac{\partial}{\partial u}$
$\frac{\gamma}{\alpha+1}(\alpha u+\beta)^{\frac{1}{\alpha}+1}$	$\sigma(\alpha u+\beta)^{3+\frac{2}{\alpha}}$	$X_3=(\alpha+1)t\frac{\partial}{\partial t}+(\alpha+1)x\frac{\partial}{\partial x}-(\alpha u+\beta)\frac{\partial}{\partial u}$
α	$-\frac{\alpha^2}{4}u+\rho$	$X_3=\frac{e^{-\frac{\alpha}{4}(t-x)}}{\alpha}\big(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}+\alpha(\beta-\frac{1}{2}u)\frac{\partial}{\partial u}\big)$ $X_4=\frac{e^{-\frac{\alpha}{4}(t+x)}}{\alpha}\big(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\alpha(\beta-\frac{1}{2}u)\frac{\partial}{\partial u}\big)$
α	$\frac{\beta}{2}u^2+\gamma u+$ $\frac{\gamma^2}{2\beta}-\frac{\alpha^4}{32\beta}$	$X_3=\frac{4}{\alpha}e^{\frac{\alpha}{4}(t+x)}\big(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}-\frac{\alpha}{4}(2u+\frac{2\gamma}{\beta}+\frac{\alpha^2}{2\beta})\big)\frac{\partial}{\partial u}$ $X_4=\frac{4}{\alpha}e^{\frac{\alpha}{4}(t-x)}\big(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}-\frac{\alpha}{4}(2u+\frac{2\gamma}{\beta}+\frac{\alpha^2}{2\beta})\big)\frac{\partial}{\partial u}$
α	$\frac{\gamma}{(\sigma+1)(2\sigma+1)}(\sigma u+\beta)^{2+\frac{1}{\sigma}}$ $-\frac{\alpha^2}{4}u-\frac{\alpha^2\beta}{4\sigma}$	$X_3=-\frac{2\sigma}{\alpha}e^{\frac{\alpha}{4\sigma}(1+\sigma)(t+x)}\big(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}-\frac{\alpha}{2\sigma}(\sigma u+\beta)\frac{\partial}{\partial u}\big)$ $X_4=-\frac{2\sigma}{\alpha}e^{\frac{\alpha}{4\sigma}(1+\sigma)(t-x)}\big(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}-\frac{\alpha}{2\sigma}(\sigma u+\beta)\frac{\partial}{\partial u}\big)$

Tab. 4.1: Cases admitting non-minimal symmetries for Eq.(4.2.1)

4.5 *Reductions & Exact Solutions*

We can use symmetries to reduce PDEs into simpler forms by introducing new similarity variables. These variables are invariant functions that can be utilized to reduce the number of variables in the considered equation by one. The invariant functions are obtained by solving the characteristic system produced by the equation $X(I)=0$, where X is a symmetry generator of the studied equation. This procedure for performing the reduction of a given PDE using its symmetries is explained in many references, e.g, [12–22].

In this section, we plan to reduce the damped wave equation (4.2.1) by one independent variable to convert it into an ODE. In this context, we use a linear combination of the two minimal symmetries of $< G4 >$ group. We also use the extra non-minimal symmetries in each case for performing the reduction.

Case 1 (**f and g are arbitrary**)

Recall that the combination of symmetries, $X = X_1 + X_2 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$, is a symmetry of Eq.(4.2.1) for arbitrary functions f and g. This means that the reduction performed using this combination is valid for all possible forms of Eq.(4.2.1). This reduction can be achieved by solving the characteristic system,

$$\frac{dt}{1} = \frac{dx}{1} = \frac{du}{0}. \tag{4.5.1}$$

Obviously, the solution of Eq.(4.5.1) is given by $t - x = k_1$ and $u = k_2$, where k_1 and k_2 are constants. Thus, we can construct the new similarity variables as,

$$\xi(t,x) = t - x, \qquad V(\xi) = u.$$

Substituting the new similarity variables in the wave equation (4.2.1) reduces it into the equation,

$$V_\xi = \frac{g(V)}{f(V)}. \tag{4.5.2}$$

Thus, we can use equation (4.5.2) to obtain exact solution for all the cases discussed in table 4.1 as shown below.

Case 2 (**$f = \alpha u + \beta$ & $g = (\alpha u + \beta)^3$**)

Solving Eq.(4.5.2) for this case, we obtain the following, traveling-wave like, solution,

$$u(t,x) = \frac{1}{\alpha c_1 - \alpha^2 \gamma (t - x)} - \frac{\beta}{\alpha}. \tag{4.5.3}$$

Furthermore, if we use, $X_3 == t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - (u + \frac{\beta}{\alpha})\frac{\partial}{\partial u}$ in reducing the damped wave equation (4.2.1) we obtain the equation,

$$(1-\zeta^2)V_{\xi\xi} + (\alpha V - 3\xi)V_\xi - 2V - \gamma\alpha^3 V^3 = 0,$$

where $\xi(t,x) = \frac{t}{x}$ and $V(\xi) = x(u + \frac{\beta}{\alpha})$.

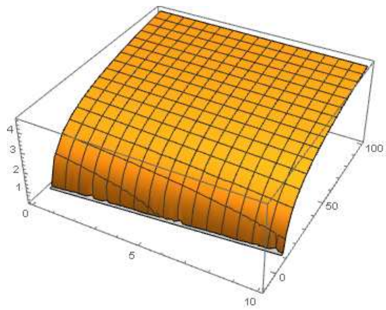

Fig. 4.1: A graph of the solution (4.5.3) for fixed values of the parameters

Case 3 $\quad$ **($f = \frac{\beta}{\alpha}e^{\alpha u}$ & $g = \sigma e^{2\alpha u}$)**

Solving Eq.(4.5.2) leads again to a traveling wave-like solution of the form,

$$u(t,x) = -\frac{1}{\alpha}log\big|c_1 - \frac{\alpha^2\sigma}{\beta}(t-x)\big|. \qquad (4.5.4)$$

Moreover, $X_3 = \alpha t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} - \frac{\partial}{\partial u}$, can be utilized to reduce Eq.(4.2.1) to the equation,

$$(1-\xi^2)V_{\xi\xi} + (\frac{\beta}{\alpha}e^{\alpha V} - 2\xi)V_\xi - \sigma e^{2\alpha V} - \frac{1}{\alpha^2} = 0,$$

where $\xi(t,x) = \frac{t}{x}$ and $V(\xi) = u - \frac{1}{\alpha}log|x|$.

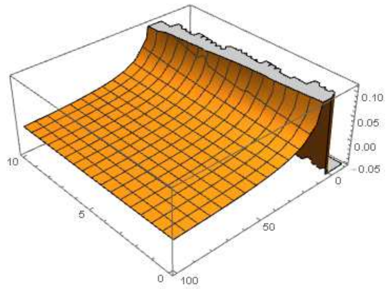

Fig. 4.2: A graph of the solution (4.5.4) for fixed values of the parameters

Case 4 $\quad (f = \frac{\gamma}{\alpha+1}(\alpha u+\beta)^{\frac{1}{\alpha}+1}$ **&** $g = \sigma(\alpha u+\beta)^{3+\frac{2}{\alpha}})$

Solving Eq.(4.5.2) for this particular case provides the following traveling wave-like exact solution:

$$u(t,x) = \frac{1}{\alpha}(c_1 - \frac{\sigma(\alpha+1)^2(t-x)}{\gamma})^{-\frac{\alpha}{\alpha+1}} - \frac{\beta}{\alpha}. \tag{4.5.5}$$

In addition, if we use the symmetry, $X_3 = (\alpha+1)t\frac{\partial}{\partial t} + (\alpha+1)x\frac{\partial}{\partial x} - (\alpha u+\beta)\frac{\partial}{\partial u}$, in reducing Eq.(4.2.1), then we obtain the following ODE:

$$\frac{\gamma}{\alpha^2}VV_\xi V_{\xi\xi} = \frac{2\alpha+1}{\alpha+1}V + \xi V_\xi + \frac{3\alpha+2}{\alpha}\xi V_\xi + \frac{\xi^2}{\alpha}(\alpha+1)V_{\xi\xi} + \sigma(\alpha+1)V = 0.$$

where $\xi(t,x) = \frac{t}{x}$ and $V(\xi) = (\alpha u+\beta)x^{\frac{\alpha}{\alpha+1}}$.

Case 5 $\quad (f = \alpha$ **&** $g = -\frac{\alpha^2}{4}u + \rho)$

Obviously, for this case we can integrate (4.5.2) to obtain the following exact solution:

$$u(t,x) = -\frac{4c_1}{\alpha^2}e^{-\frac{\alpha}{4}(t-x)} + \frac{4\rho}{\alpha^2}. \tag{4.5.6}$$

Moreover, reducing Eq.(4.2.1) using either X_3 or X_4, leads to the following identity:

$$\rho = \frac{\beta\alpha^2}{2}.$$

Case 6 $\quad (f = \alpha \text{ \& } g = \frac{\beta}{2}u^2 + \gamma u + \frac{\gamma^2}{2\beta} - \frac{\alpha^4}{32\beta})$

Thus, the solution of Eq.(4.5.2) for the derived values of $f(u)$ and $g(u)$ is given by,

$$u(t,x) = \frac{\alpha^2(1 + c_1 e^{\frac{\alpha}{4}(t-x)})}{4\beta(1 - c_1 e^{\frac{\alpha}{4}(t-x)})} - \frac{\gamma}{\beta}. \tag{4.5.7}$$

It is interesting to note that using either X_3 or X_4 in reducing Eq.(4.2.1), leads to the simple reduced algebraic equation $V = 0$ where,

$$V(\xi) = (u + \frac{\gamma}{\beta} + \frac{\alpha^2}{4\beta})e^{\pm\frac{\alpha}{2}x}.$$

Consequently, we obtain the trivial solution of u being constant, or explicitly,

$$u(t,x) = -\frac{\alpha^2}{4\beta} - \frac{\gamma}{\beta}. \tag{4.5.8}$$

Case 7 $\quad (f = \alpha \text{ \& } g = \frac{\gamma}{(\sigma+1)(2\sigma+1)}(\sigma u + \beta)^{2+\frac{1}{\sigma}} - \frac{\alpha^2}{4}u - \frac{\alpha^2\beta}{4\sigma})$

It turns out that solving Eq.(4.5.2) for this case, reduces Eq.(4.2.1) into the following ODE:

$$V_\xi + \frac{\alpha}{4}V - \frac{\gamma}{\alpha(\sigma+1)(2\sigma+1)}(\sigma V + \beta)^{2+\frac{1}{\sigma}} + \frac{\alpha\beta}{4\sigma} = 0,$$

where $\xi(t,x) = t - x$ and $V(\xi) = u$. As in the previous case, either of the symmetries X_3 or X_4 reduces the wave equation into the simple equation $V = 0$ where,

$$V(\xi) = (\sigma u + \beta)e^{\pm\frac{\alpha}{2}x}.$$

As a result, we obtain again a constant solution given by,

$$u(t,x) = -\frac{\beta}{\sigma}. \tag{4.5.9}$$

A list of obtained solutions and reduced equations for the damped wave equations (4.2.1) are presented in table 4.2 and 4.3.

4.6 ***Conclusion***

In this chapter, we have carried out a symmetry classification of a class of 1-dimensional nonlinear damped wave equation. It has been shown that the minimal subalgebra admitted by the equation is two dimensional. In some interesting cases, we have shown that the subalgebra can be extended by one to two extra symmetries. These symmetries have been utilized in performing different similarity reductions and, where possible, exact solutions have been obtained. Although most of the solutions obtained are traveling-wave like solutions, one may obtain different forms of solutions in cases where the derived equations could not be solved.

The study presented in this chapter paves the way for further investigation regarding nonlinear damped wave equations arising in mathematical physics or other scientific fields. In particular the nonlinear damped wave equations in higher dimensions could also be considered as we will do in the next chapter.

$f(u)$	$g(u)$	$u(t,x)$
$\alpha u+\beta$	$\gamma(\alpha u+\beta)^3$	$\frac{1}{\alpha c_1-\alpha^2\gamma(t-x)}-\frac{\beta}{\alpha}$
$\frac{\beta}{\alpha}e^{\alpha u}$	$\sigma e^{2\alpha u}$	$-\frac{1}{\alpha}\ln\lvert-\frac{\sigma\alpha^2}{\beta}(t-x)+c_1\rvert$
$\frac{\gamma}{\alpha+1}(\alpha u+\beta)^{1+\frac{1}{\alpha}}$	$\sigma(\alpha u+\beta)^{3+\frac{2}{\alpha}}$	$\frac{1}{\alpha}(c_1-\frac{\sigma(\alpha+1)^2(t-x)}{\gamma})^{-\frac{\alpha}{\alpha+1}}-\frac{\beta}{\alpha}$
α	$-\frac{\alpha^2}{4}u+\rho$	$-\frac{4c_1}{\alpha^2}e^{-\frac{\alpha}{4}(t-x)}+\frac{4\rho}{\alpha^2}$
α	$\frac{\beta}{2}u^2+\gamma u+\frac{\gamma^2}{2\beta}-\frac{\alpha^4}{32\beta}$	$-\frac{\gamma}{\beta}-\frac{\alpha^2}{4\beta}$
α	$\frac{\gamma}{(\sigma+1)(2\sigma+1)}(\sigma u+\beta)^{2+\frac{1}{\sigma}}-\frac{\alpha^2}{4}u-\frac{\alpha^2\beta}{4\sigma}$	$-\frac{\beta}{\sigma}$

Tab. 4.2: Some exact solutions of wave eq.(4.2.1) with particular forms of f and g

Case No.	Symmetry	Reduced equation
1	X_1	$V_{\xi\xi} = -g(V)$
	X_2	$V_{\xi\xi} + f(V)V_\xi = g(V)$
	$X_1 + X_2$	$V_\xi = \frac{g(V)}{f(V)}$
2	$X_1 + X_2$	$\alpha V + \beta = \frac{1}{c_1 - \alpha\gamma(t-x)}$
	X_3	$(1-\xi^2)V_{\xi\xi} + (\alpha V - 3\xi)V_\xi - 2V - \gamma\alpha^3 V^3 = 0$
3	$X_1 + X_2$	$V_\xi = \frac{\alpha\sigma}{\beta} e^{\alpha V}$
	X_3	$(1-\xi^2)V_{\xi\xi} + (\frac{\beta}{\alpha} e^{\alpha V} - 2\xi)V_\xi - \sigma e^{2\alpha V} - \frac{1}{\alpha^2} = 0$
4	$X_1 + X_2$	$V_\xi = \frac{\sigma(\alpha+1)}{\gamma}(\alpha V + \beta)^{2+\frac{1}{\alpha}}$
	X_3	$\frac{\gamma}{\alpha^2} V V_\xi V_{\xi\xi} = \frac{2\alpha+1}{\alpha+1} V + (\xi + \frac{3\alpha+2}{\alpha}\xi)V_\xi + \frac{\xi^2(\alpha+1)}{\alpha} V_{\xi\xi} + \sigma(\alpha+1)V$
5	$X_1 + X_2$	$V_\xi = -\frac{\alpha}{4} V + \frac{\gamma}{\alpha}$
	X_3	$\rho = \frac{\beta\alpha^2}{2}$
	X_4	$\rho = \frac{\beta\alpha^2}{2}$
6	$X_1 + X_2$	$V_\xi = \frac{\beta}{2\alpha} V^2 + \frac{\gamma}{\alpha} V + \frac{\gamma^2}{2\alpha\beta} - \frac{\alpha^3}{32\beta}$
	X_3	$V = 0$
	X_4	$V = 0$
7	$X_1 + X_2$	$V_\xi + \frac{\alpha}{4} V - \frac{\gamma}{\alpha(\sigma+1)(2\sigma+1)}(\sigma V + \beta)^{2+\frac{1}{\sigma}} + \frac{\alpha\beta}{4\sigma} = 0$
	X_3	$V = 0$
	X_4	$V = 0$

Tab. 4.3: Some Reductions of wave eq.(4.2.1)

5. CLASSIFICATION OF SYMMETRIES OF THE MULITDIMENSIONAL NONLINEAR DAMPED WAVE EQUATIONS

In this chapter, we extend our investigation of the damped wave equations to higher dimensional wave equations with a variable damping. We give a brief introduction in the first section then we formulate our problem in the second section. The derivation of the determining equations is presented in section 5.3. In section 5.4, we solve the determining system in order to perform symmetry classification of the considered damped wave equation. Section 5.5 is devoted to performing reductions and obtaining solutions of the studied equation. Finally, we give a brief discussion of the results in section 5.6.

5.1 ***Introduction***

Generally, the group classification problems are applied when studying PDEs involving arbitrary functions. As we have seen in the previous chapter, these types of problems are addressed by achieving two main steps. The first step is to find the Lie symmetries of the given PDE in its most general form, i.e with its functions being arbitrary. This step gives rise to the minimal Lie algebra. The second step is to determine all possible forms of the functions that admit larger symmetry algebra.

In chapter 4, we have examined the effect of coupling a damping term on the symmetry properties of the one dimensional wave equation with a source term.

This investigation could be extended to higher dimensional wave equations. In particular, if we add a damping term with variable damping coefficient to the following wave equation:

$$u_{tt} = div(g(u)grad\ u), \tag{5.1.1}$$

then we expect this damping term to change its symmetry properties. This fact can be shown by carrying out a symmetry classification of the considered damped wave equation. The classification can be performed using the same procedure that has been implemented in chapter 4.

5.2 *Problem Formulation*

The aim of this chapter is to extend the investigation of damped wave equations by considering the (2+1) nonlinear damped wave equation given by,

$$u_{tt} + f(u)u_t = div(g(u)grad\ u), \tag{5.2.1}$$

where u is a function of t, x and y. The studied equation is classified in terms of Lie point symmetries it admits. These symmetries are utilized in performing different similarity reductions and, where possible, exact solutions are obtained.

5.3 *Deriving the Determining Equations*

We consider Eq.(5.2.1), which can be equivalently written in the following form:

$$u_{tt} + f(u)u_t = g(u)(u_{xx} + u_{yy}) + g'(u)(u_x^2 + u_y^2). \tag{5.3.1}$$

To obtain the symmetry algebra of Eq.(5.3.1), we take the infinitesimal generator of symmetry algebra of the form,

$$X = \xi\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}\ , \tag{5.3.2}$$

where the coefficients ξ, γ, τ and ϕ are functions of x, y, t and u. Using the invariance condition, i.e., applying the 2nd prolongation $X^{[2]}$ to Eq.(5.3.1), yields the following system of determining equations:

$$\xi_u = \gamma_u = \tau_u = \phi_{uu} = 0, \tag{5.3.3}$$

$$\xi_t - g(u)\tau_x = 0, \tag{5.3.4}$$

$$\xi_y + \gamma_x = 0, \tag{5.3.5}$$

$$\gamma_t - g(u)\tau_y = 0, \tag{5.3.6}$$

$$f_u\phi + f(u)\tau_t - \tau_{tt} + g(u)\tau_{xx} + g(u)\tau_{yy} + 2\phi_{tu} = 0, \tag{5.3.7}$$

$$-f(u)\gamma_t - \gamma_{tt} + g(u)\gamma_{xx} + g(u)\gamma_{yy} - 2g(u)\phi_{yu} = 0, \tag{5.3.8}$$

$$-f(u)\xi_t - \xi_{tt} + g(u)\xi_{xx} + g(u)\xi_{yy} - 2g(u)\phi_{xu} = 0, \tag{5.3.9}$$

$$f(u)\phi_t + \phi_{tt} - g(u)\phi_{xx} - g(u)\phi_{yy} = 0, \tag{5.3.10}$$

$$-g_u\phi + 2g(u)\xi_x - 2g(u)\tau_t = 0, \tag{5.3.11}$$

$$-g_u\phi + 2g(u)\gamma_y - 2g(u)\tau_t = 0. \tag{5.3.12}$$

5.4 *Classification of Symmetries*

Since we are interested in classifying Lie symmetries and corresponding solutions of Eq.(5.3.1), we start by writing Eq.(5.3.12) in the form,

$$\frac{g_u}{g}\phi = 2\gamma_y - 2\tau_t. \tag{5.4.1}$$

From above equation, two cases are considered. Namely, $\phi = 0$ and $\phi \neq 0$. We discuss each case separately as follows.

Case I $(\phi = 0)$

Substituting $\phi = 0$ in the over-determined system, Eqs.(5.3.3)-(5.3.12), leads to the following solution:

$$\tau = c_1, \quad \xi = c_2 - c_4 y, \quad \gamma = c_3 + c_4 x.$$

The above solution is valid for arbitrary f and arbitrary g, and consequently leads to the minimal subalgebra group $< G4 >$ of vector fields $< \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} >$.

Case II $(\phi \neq 0)$

In this case, we can re-consider Eq.(5.4.1) by writing it in the form,

$$\frac{g_u}{g} = \frac{2\gamma_y - 2\tau_t}{\phi} . \tag{5.4.2}$$

Since the left hand side of Eq.(5.4.2) depends only on u, it can be put in the form,

$$\frac{g_u}{g} = \frac{2\gamma_y - 2\tau_t}{\phi} = h(u). \tag{5.4.3}$$

At this stage, we consider three possibilities, viz., $h(u) = 0, h(u) = constant \neq 0$ and $h(u)$ functionally depending on u.

Case II-A $(h(u) = 0)$

In this case, Eq.(5.4.3) can be put in the form,

$$\frac{g_u}{g} = \frac{2\gamma_y - 2\tau_t}{\phi} = 0. \tag{5.4.4}$$

From above equation it is instantly found that $g(u) = \alpha$, while $\gamma_y = \tau_t$. Substituting for g in Eq.(5.3.9) and differentiating the resulting equation with respect to u, gives

$$f_u \xi_t = 0. \tag{5.4.5}$$

Thus, three cases arise from Eq.(5.4.5). Namely, 1) $f_u \neq 0$, $\xi_t = 0$, 2)$f_u = 0$, $\xi_t \neq 0$ and 3)$f_u = \xi_t = 0$, which we consider one by one.

Case II-A-1 $(f_u \neq 0\ ,\ \xi_t = 0)$

Differentiating Eq.(5.3.7) with respect to u gives,

$$f_{uu}\phi + f_u \phi_u + f_u \tau_t = 0. \tag{5.4.6}$$

Since $f_u \neq 0$, the above equation can be written as,

$$\frac{f_{uu}}{f_u} = -\frac{\phi_u + \tau_t}{\phi} = G(u). \tag{5.4.7}$$

Notice that Eq.(5.4.7) generates three possibilities; $G(u) = 0$, $G(u) = constant \neq 0$ and $G'(u) \neq 0$.

Case II-A-1-a $(G(u) = 0)$

Hence, Eq.(5.4.7) becomes,

$$\frac{f_{uu}}{f_u} = -\frac{\phi_u + \tau_t}{\phi} = 0. \tag{5.4.8}$$

Solving Eq(5.4.8) and substituting the solution in the over-determined system, (Eqs.(5.3.3)-(5.3.12)), yields the following solution:

$$\tau = c_1 + c_5 t, \quad \xi = c_2 + c_5 x - c_4 y, \quad \gamma = c_3 + c_4 x + c_5 y, \quad \phi = -c_5(\frac{\sigma}{\beta} + u),$$

with $f(u) = \beta u + \sigma$ such that $\beta \neq 0$ and $u > -\frac{\sigma}{\beta}$. The generators of the five dimensional symmetry group is generated by vector fields given by $< \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - (\frac{\sigma}{\beta} + u)\frac{\partial}{\partial u} >$.

Case II-A-1-b $\;(G(u) = \beta \neq 0)$

In this case, Eq(5.4.7) takes the form,

$$\frac{f_{uu}}{f_u} = -\frac{\phi_u + \tau_t}{\phi} = \beta. \tag{5.4.9}$$

Following the method adopted in case II-A-1-a, leads to the following solution of the over-determined system Eqs.(5.3.3)-(5.3.12):

$$\tau = c_1 - c_5 t, \quad \xi = c_2 - c_5 x - c_4 y, \quad \gamma = c_3 + c_4 x - c_5 y, \quad \phi = \frac{c_5}{\beta},$$

with $f(u) = \frac{\sigma}{\beta} e^{\beta u}$ such that $\sigma \neq 0$ and $\frac{\sigma}{\beta} > 0$. The corresponding algebra of the 5−symmetry group is spanned by $< \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, -t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \frac{1}{\beta}\frac{\partial}{\partial u} >$.

Case II-A-1-c $\;(G'(u) \neq 0)$

To deal with this case, we first differentiate Eq(5.4.7) with respect to u to obtain,

$$G(u)\phi_u + G'(u)\phi = 0. \tag{5.4.10}$$

It is straightforward to solve the above equation to get,

$$\phi = \frac{A(t, x, y)}{G(u)}. \tag{5.4.11}$$

Substituting Eq(5.4.11) in Eq.(5.3.3) yields,

$$G(u) = \frac{1}{\beta u + \rho}. \tag{5.4.12}$$

Hence, Eq.(5.4.11) becomes,

$$\phi = A(t, x, y)(\beta u + \rho). \tag{5.4.13}$$

Also, Eq.(5.4.7) can be re-cast as,

$$\frac{f_{uu}}{f_u} = \frac{1}{\beta u + \rho} \ . \tag{5.4.14}$$

Solving above equation immediately yields,

$$f(u) = \frac{\sigma}{\beta + 1}(\beta u + \rho)^{1+\frac{1}{\beta}}, \tag{5.4.15}$$

where $\beta \notin \{0, -1\}$, $\frac{\sigma}{\beta+1} > 0$ and $u > -\frac{\rho}{\beta}$. After some more manipulation, the following solution of the over-determined system, Eqs.(5.3.3)-(5.3.12), is obtained:

$$\tau = c_1 + c_5 t, \quad \xi = c_2 + c_5 x - c_4 y, \quad \gamma = c_3 + c_4 x + c_5 y, \quad \phi = -\frac{c_5}{\beta + 1}(\beta u + \rho),$$

with $g(u) = \alpha$. Accordingly, the vector fields generating a five parameter Lie group are given by, $< \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, t^{\partial}\partial t + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{\beta+1}(\beta u + \rho)\frac{\partial}{\partial u} >$

.

Case II-A-2 $(f_u = 0 \ , \ \xi_t \neq 0)$

In this case, $f(u) = \beta$. Hence, since both functions f and g are constant, then we expect this case to give the maximal set of Lie symmetries. After carrying out some calculations, it turns out that the obtained solution of the determining system is:

$$\tau = c_1 + 2c_5 x + 2c_6 y, \quad \xi = c_2 + 2c_5\alpha t - c_4 y, \quad \gamma = c_3 + 2c_6\alpha t + c_4 x,$$

$$\phi = (c_7 - c_5\beta x - c_6\beta y)u + A(t,x,y),$$

where, $A(t,x,y)$ satisfies the differential constraint given by,

$$\beta A_t + A_{tt} - \alpha A_{xx} - \alpha A_{yy} = 0. \tag{5.4.16}$$

As a result, this case admits the 7−group of Lie symmetries given by,
$< \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, 2x\frac{\partial}{\partial t} + 2\alpha t\frac{\partial}{\partial x} - \beta x u\frac{\partial}{\partial u}, 2y\frac{\partial}{\partial t} + 2\alpha t\frac{\partial}{\partial y} - \beta y u\frac{\partial}{\partial u}, u\frac{\partial}{\partial u} >$
along with an arbitrary symmetry $A(t,x,y)\frac{\partial}{\partial u}$.

Case II-A-3 $(f_u = \xi_t = 0)$

It turns out that this case leads to six finite dimensional symmetry algebra which is a subset of the algebra obtained in case II-A-2.

Case II-B $(h(u) = constant \neq 0)$

In this case, Eq.(5.4.3) takes the form,

$$\frac{g_u}{g} = \frac{2\gamma_y - 2\tau_t}{\phi} = \alpha, \tag{5.4.17}$$

which can be easily integrated over u to give,

$$g(u) = \beta e^{\alpha u}. \tag{5.4.18}$$

Also, solving Eq.(5.4.17) for ϕ yields,

$$\phi = A(t,x,y). \tag{5.4.19}$$

Using Eq.(5.4.19) in the over-determined system, Eqs.(5.3.3)-(5.3.12), and after manipulating further leads to the following solution:

$$\tau = c_1 + c_5 t, \quad \xi = c_2 - c_4 y + c_5 (1 - \frac{\alpha}{2\sigma}) x, \quad \gamma = c_3 + c_4 x + c_5 (1 - \frac{\alpha}{2\sigma}) y, \quad \phi = -\frac{c_5}{\sigma},$$

subject to $f(u) = \frac{\rho}{\sigma} e^{\sigma u}$ where $\sigma \notin \{0, \frac{\alpha}{2}\}$ and $\frac{\rho}{\sigma} > 0$. The symmetry algebra for this case is generated by the following five vector fields $< \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, t\frac{\partial}{\partial t} + (1 - \frac{\alpha}{2\sigma}) x \frac{\partial}{\partial x} + (1 - \frac{\alpha}{2\sigma}) y \frac{\partial}{\partial y} - \frac{1}{\sigma}\frac{\partial}{\partial u} >$.

Case II-C $(h'(u) \neq 0)$

In this case, we write Eq.(5.4.3) as,

$$\phi h(u) = 2\gamma_y - 2\tau_t. \tag{5.4.20}$$

Differentiating the above equation with respect to u and solving immediately gives,

$$\phi = \frac{A(t, x, y)}{h(u)} , \tag{5.4.21}$$

where $A(t, x, y)$ is an integration function. Substituting Eq.(5.4.21) in Eq.(5.3.3) gives,

$$h(u) = \frac{1}{\alpha u + \beta} . \tag{5.4.22}$$

Consequently, Eq.(5.4.21) becomes,

$$\phi = A(t, x, y)(\alpha u + \beta). \tag{5.4.23}$$

After some more manipulations, Eq(5.4.23) reduces to,

$$\phi = A(t)(\alpha u + \beta). \tag{5.4.24}$$

Using Eq.(5.4.22) in Eq.(5.4.3) gives,

$$\frac{g_u}{g} = \frac{1}{\alpha u + \beta} \ , \tag{5.4.25}$$

which on integration over u becomes,

$$g(u) = \lambda(\alpha u + \beta)^{\frac{1}{\alpha}}. \tag{5.4.26}$$

Furthermore, simple calculations show that τ is a function of t only. At this point, substituting Eq.(5.4.24) in Eq.(5.3.7) and differentiating the resulting equation with respect to u we obtain,

$$f_{uu}(\alpha u + \beta)A(t) = (-\alpha A(t) - \tau_t)f_u. \tag{5.4.27}$$

At this stage we consider two possibilities, viz., $f_u \neq 0$ and $f_u = 0$.

Case II-C-1 $(f_u \neq 0)$

After some manipulations, the following solution is obtained:

$$\tau = c_1 - \frac{2c_5(\alpha+\sigma)}{1-2(\alpha+\sigma)}t, \quad \xi = c_2 - c_4 y + c_5 x, \quad \gamma = c_3 + c_4 x + c_5 y,$$

$$\phi = \frac{2c_5}{1-2(\alpha+\sigma)}(\alpha u + \beta),$$

where $f(u) = \frac{\rho}{\sigma+\alpha}(\alpha u + \beta)^{\frac{\sigma}{\alpha}+1}$, provided the following restrictions are met: $\sigma + \alpha \neq 0, \alpha \neq 0$ and $u > -\frac{\beta}{\alpha}$.

As a result, the symmetry algebra for this case is constructed, generating a five parameter group,

$< \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, -\frac{2(\alpha+\sigma)}{1-2(\alpha+\sigma)}t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{2(\alpha u+\beta)}{1-2(\alpha+\sigma)}\frac{\partial}{\partial u} > .$

Case II-C-2 $(f'(u) = 0)$

This case suggests that $f(u) = \sigma > 0$. Then, carrying out further calculations, one arrives at the following results:

$$\tau = (c_5 - \frac{c_7}{2})t - \frac{c_6}{2\sigma^2}e^{-\sigma t} + c_1, \quad \xi = c_5 x - c_4 y + c_2, \quad \gamma = c_4 x + c_5 y + c_3, \tag{5.4.28}$$

$$\phi = (c_7 - \frac{c_6}{\sigma}e^{-\sigma t})(\alpha u + \beta). \tag{5.4.29}$$

Substituting Eqs.(5.4.28) and (5.4.29) in Eq.(5.3.7) yields,

$$\sigma(c_5 - \frac{c_7}{2}) + c_6 e^{-\sigma t}(1 + 2\alpha) = 0. \tag{5.4.30}$$

It is obvious that Eq.(5.4.30) leads to the following two equations,

$$c_7 = 2c_5, \tag{5.4.31}$$

$$c_6(1 + 2\alpha) = 0. \tag{5.4.32}$$

As a result, Eq.(5.4.32) gives rise to two possibilities. Namely, $\alpha = -1/2$ or $c_6 = 0$.

Case II-C-2-a $(\alpha = -1/2, \ c_6 \neq 0)$

Substituting Eq.(5.4.31) and the value of α in Eqs.(5.4.28) and (5.4.29) gives,

$$\tau = c_1 - \frac{c_6}{2\sigma^2}e^{-\sigma t}, \qquad \xi = c_2 - c_4 y + c_5 x, \qquad \gamma = c_3 + c_4 x + c_5 y,$$

$$\phi = (2c_5 - \frac{c_6}{\sigma}e^{-\sigma t})(\beta - \frac{1}{2}u).$$

Consequently, the symmetry algebra is extended by two extra symmetries given as,

$< \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \ x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2(\beta - \frac{1}{2}u)\frac{\partial}{\partial u}, \ -\frac{e^{-\sigma t}}{2\sigma^2}\frac{\partial}{\partial t} - \frac{e^{-\sigma t}}{\sigma}(\beta - \frac{1}{2}u)\frac{\partial}{\partial u} >,$

where $\sigma \neq 0$. Notice also that Eq.(5.4.26) in this case takes the form,

$$g(u) = \lambda(\beta - \frac{1}{2}u)^{-2}.$$

Case II-C-2-b $(\alpha \neq -1/2,\ c_6 = 0)$

The solution of the over-determined system for this case takes the form,

$$\tau = c_1, \quad \xi = c_2 - c_4 y + c_5 x, \quad \gamma = c_3 + c_4 x + c_5 y, \quad \phi = 2c_5(\alpha u + \beta).$$

which generates a five dimensional symmetry group,
$< \frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2(\alpha u + \beta)\frac{\partial}{\partial u} >$.
Finally, our calculations have shown that the case $\alpha = -1/2$ with $c_6 = 0$ is not interesting as it does not generate extra symmetries.
At this stage, we conclude section 5.4 where it is shown that Eq.(5.3.1) has at least four dimensional symmetry algebra which is given by the group $< G4 >$. In this section we have presented the particular forms of f and g that admit extra symmetries. The table 5.1 below shows the extra symmetries obtained for each case arising in the classification.

Case no.	$f(u)$	$g(u)$	Symmetries other than minimal
1	arbitrary	arbitrary	No extra symmetries
2	$\beta u+\sigma$	α	$X_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - (\frac{\sigma}{\beta}+u)\frac{\partial}{\partial u}$
3	$\frac{\sigma}{\beta}e^{\beta u}$	α	$X_5 = -t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \frac{1}{\beta}\frac{\partial}{\partial u}$
4	$\frac{\sigma}{\beta+1}(\beta u+\rho)^{1+\frac{1}{\beta}}$	α	$X_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{\beta+1}(\beta u+\rho)\frac{\partial}{\partial u}$
5	β	α	$X_5 = 2x\frac{\partial}{\partial t} + 2\alpha t\frac{\partial}{\partial x} - \beta xu\frac{\partial}{\partial u}$ $X_6 = 2y\frac{\partial}{\partial t} + 2\alpha t\frac{\partial}{\partial x} - \beta yu\frac{\partial}{\partial u}$ $X_7 = u\frac{\partial}{\partial u}$ $X_\infty = A(t,x,y)\frac{\partial}{\partial u}$
6	$\frac{\rho}{\sigma}e^{\sigma u}$	$\beta e^{\alpha u}$	$X_5 = t\frac{\partial}{\partial t} + (1-\frac{\alpha}{2\sigma})x\frac{\partial}{\partial x} + (1-\frac{\alpha}{2\sigma})y\frac{\partial}{\partial y} - \frac{1}{\sigma}\frac{\partial}{\partial u}$
7	$\frac{\rho}{\alpha+\sigma}(\alpha u+\beta)^{\frac{\sigma}{\alpha}+1}$	$\lambda(\alpha u+\beta)^{\frac{1}{\beta}}$	$X_5 = \frac{-2(\alpha+\sigma)}{1-2(\alpha+\sigma)}t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{2(\alpha u+\beta)}{1-2(\alpha+\sigma)}\frac{\partial}{\partial u}$
8	σ	$\lambda(\alpha u+\beta)^{\frac{1}{\alpha}}$	$X_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2(\alpha u+\beta)\frac{\partial}{\partial u}$
9	σ	$\lambda(\beta-0.5u)^{-2}$	$X_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2(\beta-0.5u)\frac{\partial}{\partial u}$ $X_6 = \frac{-e^{-\sigma t}}{2\sigma^2}\frac{\partial}{\partial t} - \frac{e^{-\sigma t}}{\sigma}(\beta-0.5u)\frac{\partial}{\partial u}$

Tab. 5.1: Non-minimal symmetries of Eq.(5.3.1) for particular forms of f and g

5.5 *Reductions & Exact Solutions of the Damped Wave Equation*

In this section, we briefly discuss solutions of the considered damped wave equation (5.3.1) by reduction via symmetry algebra. In particular, we use two dimensional subalgebras to reduce the considered equation to an ODE. In the context of this chapter, the procedure of performing the reduction is shown in details for the general form of Eq.(5.3.1), where the functions $f(u)$ and $g(u)$ are arbitrary. For particular forms of $f(u)$ and $g(u)$, the reduced equations and the corresponding similarity variables are given in table 5.3.

The commutation relations between elements of $< G4 >$ is given in table 5.2 below.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	X_3
X_3	0	0	0	$-X_2$
X_4	0	$-X_3$	X_2	0

Tab. 5.2: Commutator table for symmetries of $< G4 >$

It is obvious from table 5.2 that $[X_1, X_2] = [X_1, X_3] = [X_1, X_4] = [X_2, X_3] = 0$. Thus, we have four subalgebras of two dimensions. We perform reduction for each subalgebra as follows.

a) Reduction under the subalgebra $\langle X_1, X_2 \rangle$

The first level of reduction is performed using the symmetry, $X_1 = \frac{\partial}{\partial t}$, which gives the following characteristic system:

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dy}{0} = \frac{du}{0}. \tag{5.5.1}$$

Solving the characteristic system of Eqs.(5.5.1), it is straightforward to find the following similarity variables:

$$\xi_1(t, x, y) = x, \qquad \xi_2(t, x, y) = y, \qquad V(\xi_1, \xi_2) = u. \tag{5.5.2}$$

In the light of these similarity transformations, Eq.(5.3.1) is easily reduced to the form,

$$g(V)(V_{\xi_1\xi_1} + V_{\xi_2\xi_2}) + g'(V)(V_{\xi_1}^2 + V_{\xi_2}^2) = 0. \tag{5.5.3}$$

The second level of reduction is performed using the symmetry, $X_2 = \frac{\partial}{\partial x}$. Hence, the symmetry,

$$X = X_2(\xi_1)\frac{\partial}{\partial \xi_1} + X_2(\xi_2)\frac{\partial}{\partial \xi_2} + X_2(V)\frac{\partial}{\partial V}\ , \tag{5.5.4}$$

is inherited by Eq.(5.5.3). This inherited symmetry is used to obtain the new similarity variables. Notice that, Eq.(5.5.4) can be equivalently written as,

$$X = \frac{\partial}{\partial \xi_1} + 0\frac{\partial}{\partial \xi_2} + 0\frac{\partial}{\partial V} \ . \tag{5.5.5}$$

Thus, we solve the characteristic system generated by Eq.(5.5.5) to obtain the following similarity variables:

$$s(\xi_1, \xi_2) = \xi_2, \qquad W(s) = V. \tag{5.5.6}$$

Using the new similarity variables, Eq.(5.5.3) is reduced to,

$$g(W)W_{ss} + g'(W)W_s^2 = 0. \tag{5.5.7}$$

b) Reduction under the subalgebra$\langle X_1, X_3 \rangle$

Following the same procedure as in case (a), we reduce Eq.(5.3.1) again into Eq.(5.5.7) but with the following similarity variables, $s = x$ and $W = u$.

c) Reduction under the subalgebra $\langle X_1, X_4 \rangle$

The damped wave equation, Eq.(5.3.1), in this case is reduced to,

$$g(W)(W_s + sW_{ss}) + s^2 g'(W)W_s^2 = 0. \tag{5.5.8}$$

The corresponding similarity variables are, $s = x^2 + y^2$ and $W = u$.

d) Reduction under the subalgebra $\langle X_2, X_3 \rangle$

In this case, Eq.(5.3.1) is reduced to,

$$W_{ss} + f(W)W_s = 0, \tag{5.5.9}$$

where $s = t$ and $W = u$.

Case	Reduced equation	s	W
2	$(2s-2s^3)W_s-(s^2+s^4)W_{ss}-2W=0$	$\frac{x}{y}$	$-x(\frac{\sigma}{\beta}+u)$
2	$(s^2-\alpha s^4)W_{ss}+(sW-2s-2\alpha s^3)W_s+2W-W^2=0$	$\frac{t}{y}$	$t(\frac{\sigma}{\beta}+u)$
2	$(4s-4\alpha s^2)W_{ss}+(2W-2-4\alpha s)W_s+\frac{2}{s}W-\frac{1}{s}W^2=0$	$\frac{t^2}{x^2+y^2}$	$t(\frac{\sigma}{\beta}+u)$
3	$(s^4+s^2)W_{ss}+2s^3W_s+1=0$	$\frac{x}{y}$	$\beta u+\log x$
3	$(s^2-\alpha s^4)W_{ss}+(\frac{\sigma}{\beta}se^W-2\alpha s^3)W_s-\frac{\sigma}{\beta}e^W+1=0$	$\frac{t}{y}$	$\log t+\beta u$
3	$(4\beta^2 s+4\beta s)W_{ss}+(6s-2\sigma s-4\alpha)W_s-\frac{\sigma}{\beta}+1=0$	$\frac{x^2+y^2}{t^2}$	$\beta u+\log t$
4	$(s^2W+s^4W)W_{ss}+(-2sW-2s^3W)W_s+2W^2-\frac{(sW_s-W)^2}{\beta+1}+\frac{s^4W_s^2}{\beta+1}=0$	$\frac{x}{y}$	$x(\beta u+\rho)^{1+\frac{1}{\beta}}$
4	$(s^2W+\alpha s^4W)W_{ss}+(\frac{\sigma s}{\beta+1}+2\alpha s^3W-2sW)W_s+2W^2-\frac{(sW_s-W)^2}{\beta+1}-\frac{\sigma}{\beta+1}W=0$	$\frac{t}{y}$	$t(\beta u+\rho)^{1+\frac{1}{\beta}}$
4	$(\frac{2\sigma s}{\beta+1}W+\frac{4\alpha s^2}{\beta}W-\frac{8\alpha(\beta+1)}{\beta}s^2W-2sW)W_s+(4s^2W-\frac{4\alpha(\beta+1)}{\beta}s^3W)W_{ss}$ $+2W^2-\frac{\sigma}{\beta+1}W^3+\frac{4\alpha}{\beta}s^3W_s^2-\frac{(2sW_s-W)^2}{\beta+1}=0$	$\frac{t^2}{x^2+y^2}$	$t(\beta u+\rho)^{1+\frac{1}{\beta}}$
5	$W_s+sW_{ss}+sW_s^2+\frac{\beta^2}{16\alpha}=0$	$y^2-\alpha t^2$	$\frac{\beta}{2}t+\log u$
5	$W_s+sW_{ss}+sW_s^2+\frac{\beta^2}{16\alpha}=0$	$y^2-\alpha t^2$	$\frac{\beta}{2}t+\log u$
6	$(1-s^2)W_{ss}-(2s+\frac{4\alpha}{s(2\sigma-\alpha)})W_s+\frac{2\alpha(1+s^2)}{2\sigma-\alpha}W_s^2+\frac{1}{s^2}(1+\frac{2\alpha}{2\sigma-\alpha})=0$	$\frac{x}{y}$	$\log x+(\sigma-\frac{\alpha}{2})u$
6	$\frac{\alpha}{2\sigma}(\frac{\alpha}{2\sigma}-1)sW_s+(\frac{\alpha}{2\sigma}-1)^2s^2W_{ss}+1+\frac{\rho}{\sigma}(1-\frac{\alpha}{2\sigma})se^WW_s$ $-\frac{\rho}{\sigma}e^W-\beta e^{\alpha W}s^4W_{ss}-2\beta e^{\alpha W}s^3W_s-\frac{\alpha\beta}{\sigma}e^{\frac{\alpha}{\sigma}W}s^4W_s^2=0$	$\frac{t^{1-\frac{\alpha}{2\sigma}}}{y}$	$\log t+\sigma u$
6	$(\frac{\alpha}{2}-2)(\frac{\alpha}{2}-3)sW_s+(\frac{\alpha}{2}-2)^2s^2W_{ss}-1+\frac{\rho}{\sigma}e^W((\frac{\alpha}{2}-2)sW_s+1)$ $-\beta e^{\frac{\alpha}{\sigma}W}(4W_s+4sW_{ss})=0$	$(x^2+y^2)t^{\frac{\alpha}{\sigma}-2}$	$\log t+\sigma u$
7	$(s^4+s^2)W_{ss}+2(s^3-s)W_s$ $+\frac{(4-4\sigma)(sW_s-W)^2}{W}+\frac{(3-2\sigma)W_s^2}{2\alpha+2\sigma-1}=0$	$\frac{x}{y}$	$x(2\alpha u+2\beta)^{\frac{2\sigma+2\alpha-1}{2\alpha}}$
8	$(4\alpha^2+2\alpha)W^2+(4\alpha s+2s^3+4s)WW_s+(s^2+s^4)WW_{ss}+\frac{s^4+s^2}{\alpha}W_s^2=0$	$\frac{x}{y}$	$x^{-2\alpha}(\alpha u+\beta)$
8	$W_{ss}+\sigma W_s-(4\lambda\alpha^2+2\alpha\lambda)W^{1+\frac{1}{\alpha}}=0$	t	$y^{-2\alpha}(\alpha u+\beta)$
8	$W_{ss}+\sigma W_s-4\lambda(\alpha^2+1)W^{1+\frac{1}{\alpha}}=0$	t	$(x^2+y^2)^{-\alpha}(\alpha u+\beta)$
9	$(sW+s^3)W_{ss}+(2+2s^2)WW_s-(2s^3+2s)W_s^2=0$	$\frac{x}{y}$	$x(\beta-\frac{1}{2}u)$
9	$(1+\sigma s)WW_s+sWW_{ss}-16\lambda=0$	t	$y(2\beta-u)$
9	$WW_{ss}+\sigma WW_s-8\lambda=0$	t	$(2\beta-u)\sqrt{x^2+y^2}$
9	$(s^2+s^4)W_{ss}+(2s^3-2s)W_s+(3s^2+3s^4)W_s^2=0$	$\frac{x}{y}$	$\log\|2\beta x-xu\|+\sigma t$
9	$W_{ss}-W_s^2=0$	y	$\sigma t+\log\|\beta-\frac{1}{2}u\|$
9	$W_s+sW_{ss}-sW_s^2=0$	x^2+y^2	$\sigma t+\log\|\beta-\frac{1}{2}u\|$

In all the reductions, x's and y's are interchangeable

Tab. 5.3: Some Reductions of the damped wave equation (5.3.1)

Since reductions performed are valid for arbitrary functions $f(u)$ and $g(u)$, we can use them to obtain exact solutions for all particular forms of f and g arising from the classification.

For example, if we consider the case where, $f(u)=\frac{\rho}{\sigma}e^{\sigma u}$ and $g(u)=\beta e^{\alpha u}$, then we can obtain exact solution for the damped wave equation (5.3.1) by first substituting for g in Eq.(5.5.8) to obtain,

$$\beta e^{\alpha W}(W_s+sW_{ss})+\alpha\beta e^{\alpha W}s^2W_s^2=0. \qquad (5.5.10)$$

Then, solving Eq.(5.5.10) we get,

$$W = c_1 \log|s| - c_1 \log|as + \frac{1}{c_1}| + c_2. \tag{5.5.11}$$

Finally, in order to express the solution in terms of the original variables of Eq.(5.3.1), we transform each similarity variable in Eq.(5.5.11) to its corresponding variable given in case (c) above, i.e., we substitute for $s = x^2 + y^2$ and $W = u$ in the solution (5.5.11). This substitution gives rise to a static exact solution of Eq.(5.3.1), given by,

$$u(t, x, y) = c_1 \log(x^2 + y^2) - c_1 \log|ax^2 + ay^2 + \frac{1}{c_1}| + c_2. \tag{5.5.12}$$

Furthermore, we can perform more reductions by considering combination of symmetries. For instance, if we use the subalgebra, $\langle \frac{\partial}{\partial t} + 2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} \rangle$, then Eq.(5.3.1) is reduced into,

$$9W_{ss} - 3f(W)W_s = 2g(W)W_{ss} + 2g'(W)W_s^2, \tag{5.5.13}$$

where $s = x + y - 3t$ and $W = u$. In particular, if we choose $f(u) = e^u$ and $g(u) = \sigma$ then Eq.(5.5.13) becomes,

$$(9 - 2\sigma)W_{ss} - 3e^W W_s = 0. \tag{5.5.14}$$

The solution of Eq.(5.5.14) is given by, $W = \log|\frac{3-\frac{2}{3}\sigma}{c_1 - s}|$. This solution is equivalent to the following solution of Eq.(5.3.1):

$$u(t, x, y) = \log|\frac{3 - \frac{2}{3}\sigma}{c_1 - x - y + 3t}|. \tag{5.5.15}$$

Furthermore, if we choose $f(u) = u$ and $g(u) = \sigma$, then Eq.(5.5.13) becomes,

$$(9 - 2\sigma)W_{ss} - 3WW_s = 0. \tag{5.5.16}$$

Solving Eq.(5.5.16), we obtain, $W = \sqrt{\frac{2c_1}{3}} \tan[\frac{\sqrt{6c_1}}{2(9-2\sigma)} s + c_2]$, which gives rise to the following exact solution of Eq.(5.3.1):

$$u(t, x, y) = \sqrt{\frac{2c_1}{3}} \tan[\frac{\sqrt{6c_1}}{2(9 - 2\sigma)}(x + y - 3t) + c_2]. \tag{5.5.17}$$

Some exact solutions for different forms of the studied equation are provided in table 5.4.

5.6 *Conclusion*

In this chapter, we carried out a symmetry classification of the (2+1) nonlinear damped wave equation. It has been shown that the minimal subalgebra admitted by the equation is four dimensional. In some interesting cases, the subalgebra can be extended by one to three extra symmetries. Reductions of the studied equation have also been performed using two dimensional subalgebras where some exact solutions have been obtained.

The study presented in this chapter paves the way for further investigation regarding nonlinear damped wave equations arising in mathematical physics or other scientific fields. In particular the (3+1) nonlinear damped wave equation could also be considered.

$f(u)$	$g(u)$	$u(t,x,y)$
$\beta u+\sigma$	α	c_1x+c_2
$\beta u+\sigma$	α	$c_1\log(x^2+y^2)+c_2$
$\beta u+\sigma$	α	$\frac{\sqrt{-\sigma^2-2\beta c_1}}{\beta}\tan[(\frac{-t}{2}-\frac{c_2}{2})\sqrt{-\sigma^2-2\beta c_1}]-\frac{\sigma}{\beta}$
$\frac{\sigma}{\beta}e^{\beta u}$	α	c_1y+c_2
$\frac{\sigma}{\beta}e^{\beta u}$	α	$c_1\log(x^2+y^2)+c_2$
$\frac{\sigma}{\beta}e^{\beta u}$	α	$-\frac{1}{\beta}\log\lvert\frac{\sigma}{c_1\beta^2}-\frac{e^{-\beta c_1(y+c_2)}}{c_1\beta^2}\rvert$
$\frac{\sigma}{\beta}e^{\beta u}$	α	$c_1\arctan\frac{x}{y}+\log\lvert\frac{x}{y}\rvert-\frac{1}{2}\log(1+\frac{x^2}{y^2})+c_2$
$\frac{\sigma}{\beta}e^{\beta u}$	α	$-\frac{1}{\beta}\log\lvert 1+\frac{\sigma}{c_1\beta^2}-\frac{e^{-\beta c_1y-\beta c_1c_2}}{c_1\beta^2}\rvert$
$\frac{\sigma}{\beta+1}(\beta u+\rho)^{1+\frac{1}{\beta}}$	α	c_1y+c_2
$\frac{\sigma}{\beta+1}(\beta u+\rho)^{1+\frac{1}{\beta}}$	α	$c_1\log(x^2+y^2)+c_2$
β	α	c_1x+c_2
β	α	$c_1\log(x^2+y^2)+c_2$
β	α	$-\frac{c_1}{\beta}e^{-\beta t}+c_2$
$\frac{\rho}{\sigma}e^{\sigma u}$	$\beta e^{\alpha u}$	$\frac{\log\lvert ay+c_1\rvert}{a}+c_2$
$\frac{\rho}{\sigma}e^{\sigma u}$	$\beta e^{\alpha u}$	$\frac{\log\lvert ax+c_1\rvert}{a}+c_2$
$\frac{\rho}{\sigma}e^{\sigma u}$	$\beta e^{\alpha u}$	$c_1\log(x^2+y^2)-c_1\log(ax^2+ay^2+\frac{1}{c_1})+c_2$
$\frac{\rho}{\sigma+\alpha}(\alpha u+\beta)^{1+\frac{\sigma}{\alpha}}$	$\lambda(\alpha u+\beta)^{\frac{1}{\alpha}}$	$\frac{(c_1(\alpha+1)(y+c_2))^{\frac{\alpha}{\alpha+1}}-\beta}{\alpha}$
$\frac{\rho}{\sigma+\alpha}(\alpha u+\beta)^{1+\frac{\sigma}{\alpha}}$	$\lambda(\alpha u+\beta)^{\frac{1}{\alpha}}$	$\frac{(c_1(\alpha+1)(x+c_2))^{\frac{\alpha}{\alpha+1}}-\beta}{\alpha}$
σ	$\lambda(\alpha u+\beta)^{\frac{1}{\alpha}}$	$\frac{1}{\alpha}[(c_1y+c_2)^{\frac{\alpha}{\alpha+1}}-\beta]$
σ	$\lambda(\alpha u+\beta)^{\frac{1}{\alpha}}$	$c_2-\frac{c_1}{\sigma}e^{-\sigma t}$
σ	$\lambda(\beta-\frac{1}{2}u)^{-2}$	$c_2-\frac{c_1}{\sigma}e^{-\sigma t}$
σ	$\lambda(\beta-\frac{1}{2}u)^{-2}$	$2\beta-\frac{2}{c_1y+c_2}$
σ	$\lambda(\beta-\frac{1}{2}u)^{-2}$	$2\beta-\frac{2}{c_1x+c_2}$
σ	$\lambda(\beta-\frac{1}{2}u)^{-2}$	$2\beta-\frac{c_2e^{-\sigma t}}{y+c_1}$
σ	$\lambda(\beta-\frac{1}{2}u)^{-2}$	$2\beta-\frac{c_2e^{-\sigma t}}{x+c_1}$
σ	$\lambda(\beta-\frac{1}{2}u)^{-2}$	$2\beta-\frac{2c_2e^{-\sigma t}}{c_1-\log(x^2+y^2)}$

Tab. 5.4: Some exact solutions of Eq.(5.3.1) with particular forms of f and g

6. SYMMETRY ANALYSIS & EXACT SOLUTIONS OF THE DAMPED WAVE EQUATION ON THE SURFACE OF THE SPHERE

The aim of this chapter is to study the damped wave equation on the surface of the sphere. In the first section, a brief survey of the literature is given. The studied problem is formulated in section 6.2. In section 6.3, we find out the symmetry structure of the damped wave equation on the sphere. In section 6.4, we present a derivation of certain exact solutions for the studied equation. Finally, the last section is devoted to giving a summary and discussion of the obtained results.

6.1 ***Introduction***

In fact, investigating the symmetry properties of fundamental equations of physics has attracted the attention of many researchers [16–18]. In particular, exploiting Lie symmetries to study wave equation in flat background metric has gained a considerable amount of attention [48–52]. This type of work has been extended to non-flat background metric in order to understand the effect of curvature on the symmetry properties of the wave equation. In this regard, Azad and Mustafa gave a complete symmetry analysis of the wave equation on sphere [53]. However, studying the damped wave equations in non-flat background metric has not been given sufficient attention. Therefore, we plan in this chapter to address such problem in order to extend our investigation of the symmetry structure of damped wave equations in different Riemannian setting.

6.2 *Problem Formulation*

In this chapter, we intend to extend the investigation of the wave behavior in non-flat background metric by studying the damped wave equation in a metric of constant curvature. In particular, the symmetry algebra of the damped wave equation on the surface of the sphere is constructed. For this purpose, we address a damped wave equation of the form,

$$u_{tt} + \alpha u_t = \triangle u.$$

Recall that the metric of the surface of the sphere is given by a second rank symmetric metric tensor,

$$g_{ij} = (1, sin^2\theta). \tag{6.2.1}$$

In the light of this metric, the Laplacian, $\Delta = \frac{1}{\sqrt{|g|}} g^{ij} \frac{\partial}{\partial x_j} (\sqrt{\mid g \mid} g^{kl} \frac{\partial u}{\partial x_l})$ takes the form,

$$\Delta u = \cot\theta u_\theta + u_{\theta\theta} + \csc^2\theta u_{\phi\phi}. \tag{6.2.2}$$

Therefore, the damped wave equation on the surface of the sphere is derived as,

$$u_{tt} + \alpha u_t = \cot\theta u_\theta + u_{\theta\theta} + \csc^2\theta u_{\phi\phi}, \tag{6.2.3}$$

where u is a function of t, θ and ϕ.

After constructing the symmetry algebra of Eq.(6.2.3), we plan to implement the obtained symmetries to perform different similarity reductions of the equation. For each reduction case, the considered wave equation is reduced into an ODE, where an exact invariant solution is derived.

6.3 *Symmetry Algebra of Eq.(6.2.3)*

Recall that the procedure of obtaining Lie symmetries of a differential equation is described in details in many references, e.g.,[12–22]. In order to obtain the symmetry algebra of Eq.(6.2.3), we construct the infinitesimal generator of the form,

$$X = \xi(\theta,\phi,t,u)\frac{\partial}{\partial\theta} + \eta(\theta,\phi,t,u)\frac{\partial}{\partial\phi} + \tau(\theta,\phi,t,u)\frac{\partial}{\partial t} + \psi(\theta,\phi,t,u)\frac{\partial}{\partial u}. \quad (6.3.1)$$

Using the invariance criterion, i.e,

$$X^{[2]}\big|_{F=0} = 0, \quad (6.3.2)$$

where, $F = u_{tt}+\alpha u_t - \cot\theta u_\theta - u_{\theta\theta} - \csc^2\theta u_{\phi\phi}$, is the left hand side of Eq.(6.2.3) and $X^{[2]}$ is the second prolongation of the symmetry generator (6.3.1). Applying the invariance criterion (6.3.2), gives rise to the following system of determining equations:

$$\psi_{uu} = 0 = \tau_u = 0 = \eta_u = 0 = \xi_u = 0, \quad (6.3.3)$$

$$\xi_t - \tau_\theta = 0, \quad (6.3.4)$$

$$\tau_t - \xi_\theta = 0, \quad (6.3.5)$$

$$\xi_\phi + \sin^2(\theta)\eta_\theta = 0, \quad (6.3.6)$$

$$\eta_t - \csc^2(\theta)\tau_\phi = 0, \quad (6.3.7)$$

$$\cot\theta\xi + \eta_\phi - \xi_\theta = 0, \quad (6.3.8)$$

$$\alpha\tau_\theta + 2\psi_{\theta u} + \xi_{\theta\theta} = 0, \quad (6.3.9)$$

$$\alpha\psi_t + \psi_{tt} - \csc^2\theta\psi_{\phi\phi} - \cot\theta\psi_\theta - \psi_{\theta\theta} = 0, \quad (6.3.10)$$

$$2\psi_{tu} + \csc^2\theta\tau_{\phi\phi} + \alpha\xi_\theta + \cot\theta\tau_\theta - \xi_{t,\theta} + \tau_{\theta\theta} = 0, \tag{6.3.11}$$

$$\alpha\tau_\phi + 2\psi_{\phi u} - \eta_{\phi\phi} - \cos\theta\sin\theta\eta_\theta + \xi_{\theta\phi} - \sin^2\theta\eta_{\theta\theta} = 0. \tag{6.3.12}$$

Solving the above determining system gives rise to the following infinitesimals:

$$\tau = c_1, \qquad \xi = c_4\cos\phi + c_5\sin\phi,$$

$$\eta = c_2 - c_4\cot\theta\sin\phi + c_5\cos\phi\cot\theta,$$

$$\psi = c_3 u + f(t, r, \theta),$$

where f is any function satisfying Eq.(6.2.3).

The six infinitesimal symmetry generators associated with the above infinitesimal are given by,

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial\phi}, \qquad X_3 = u\frac{\partial}{\partial u},$$

$$X_4 = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi},$$

$$X_5 = \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi},$$

$$X_\infty = f(t, r, \theta)\frac{\partial}{\partial u}.$$

At this stage, we construct the commutator table for the derived symmetries. Recall that the commutation relation $[X_i, X_j]$ for two symmetries X_i, X_j is defined by,

$$[X_i, X_j] = X_i(X_j(f)) - X_j(X_i(f)). \tag{6.3.13}$$

The commutation relations is shown in table 6.1 below.

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	0
X_2	0	0	0	$-X_5$	X_4
X_3	0	0	0	0	0
X_4	0	X_5	0	0	$-X_2$
X_5	0	$-X_4$	0	X_2	0

Tab. 6.1: Commutator table for the symmetries of Eq.(6.2.3)

It is clear from table 6.1 that we can pick different choices of two dimensional subalgebras to perform double reduction. In particular, we restrict our attention on the following subalgebras:

$$\chi_1 = \langle X_5, X_2 + X_4 \rangle, \qquad \chi_2 = \langle -\frac{2}{\alpha} X_1 + X_3, X_4 \rangle, \qquad \chi_3 = \langle -\frac{2}{\alpha} X_1 + X_3, X_5 \rangle,$$

$$\chi_4 = \langle aX_1 + bX_2 + cX_3, dX_1 + eX_2 + fX_3 \rangle.$$

6.4 *Reductions and Exact Solutions of the Damped Wave Equation*

In this section we perform the reductions corresponding to the four subalgebras that have been mentioned in the previous section.

6.4.1 *Reduction Under the Subalgebra* $\chi_1 = \langle X_5, X_2 + X_4 \rangle$

We can convert Eq.(6.2.3) into an ODE by performing double reduction. The first level of reduction is performed using the symmetry X_5. This symmetry gives rise to the following characteristic system:

$$\frac{dt}{0} = \frac{d\theta}{\sin\phi} = \frac{d\phi}{\cos\phi\cot\theta} = \frac{du}{0} \ . \tag{6.4.1}$$

Integrating Eq.(6.4.1) leads to the following new similarity variables:

$$\xi_1(t,\theta,\phi,u) = t, \qquad \xi_2(t,\theta,\phi,u) = \sin\theta\cos\phi, \qquad W(\xi_1,\xi_2) = u. \tag{6.4.2}$$

Now substituting the new variables in Eq.(6.2.3) reduces it into,

$$W_{\xi_1\xi_1} + \alpha W_{\xi_1} = -2\xi_2 W_{\xi_2} + (1-\xi_2^2)W_{\xi_2\xi_2}. \tag{6.4.3}$$

At this stage, we perform the second reduction by using a linear combination of symmetries, $X = X_2 + X_4$. Notice that the symmetry, $X(\xi_1)\frac{\partial}{\partial\xi_1} + X(\xi_2)\frac{\partial}{\partial\xi_2} + X(V)\frac{\partial}{\partial V}$ is inherited by Eq.(6.4.3). Since $X(\xi_1) = X(W) = 0$ and $X(\xi_2) = \cos\theta - \sin\theta\sin\phi$, then the inherited symmetry can be written as,

$$X = 0\frac{\partial}{\partial\xi_1} + (\cos\theta - \sin\theta\sin\phi)\frac{\partial}{\partial\xi_2} + 0\frac{\partial}{\partial W} \ . \tag{6.4.4}$$

Utilizing this inherited symmetry in reducing Eq.(6.4.3), gives rise to the following similarity variables:

$$y(\xi_1,\xi_2) = \xi_1, \qquad V(W) = W. \tag{6.4.5}$$

Substituting the new variables in Eq.(6.4.3), reduces it into,

$$V_{yy} + \alpha V_y = 0. \tag{6.4.6}$$

Solving Eq.(6.4.6) yields,

$$V = -\frac{K_1}{\alpha} e^{-\alpha \xi_1} + K_2, \tag{6.4.7}$$

where K_1 and K_2 are constants. Finally, we transform the similarity variables into the original variables of Eq.(6.2.3) by substituting for $y = t$ and $V = u$ in Eq.(6.4.7). This substitution gives rise to the following non-static exact solution of the studied wave equation (6.2.3):

$$u(t, \theta, \phi) = -\frac{K_1}{\alpha} e^{-\alpha t} + K_2. \tag{6.4.8}$$

The obvious solution suggests that the amplitude of the wave decreases with time as shown in figure 6.1 below.

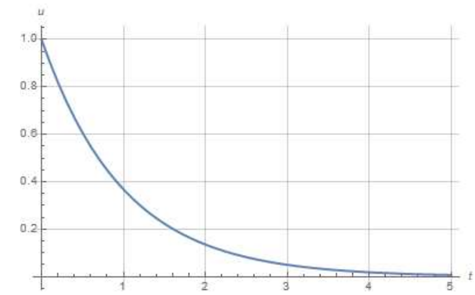

Fig. 6.1: A graph of solution (6.4.8) with fixed values of the constants

6.4.2 Reduction Under the Subalgebra $\chi_2 = \langle -\frac{2}{\alpha} X_1 + X_3, X_4 \rangle$

To solve the Eq.(6.2.3), we use the following linear combination of two symmetries,

$$X = -\frac{2}{\alpha} X_1 + X_3 = -\frac{2}{\alpha} \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}. \tag{6.4.9}$$

The characteristic system corresponding to the above symmetry leads to the following similarity variables:

$$\xi_1(t,\theta,\phi,u) = \theta, \quad \xi_2(t,\theta,\phi,u) = \phi, \quad W(\xi_1,\xi_2) = ue^{\frac{\alpha}{2}t}. \tag{6.4.10}$$

The above variables reduce Eq.(6.2.3) to,

$$-\frac{\alpha^2}{4}\sin\xi_1 W = \cos\xi_1 W_{\xi_1} + \sin\xi_1 W_{\xi_1\xi_1} + \csc\xi_1 W\xi_2\xi_2. \tag{6.4.11}$$

To perform the second level of reduction via the rotational symmetry X_4, it is noted that Eq.(6.4.11) inherits the symmetry,

$$X = \cos\xi_2\frac{\partial}{\partial\xi_1} - \cot\xi_1\sin\xi_2\frac{\partial}{\partial\xi_2} + 0\frac{\partial}{\partial W}\ . \tag{6.4.12}$$

Utilizing this inherited symmetry reduces Eq.(6.4.11) to the following Legendre equation,

$$(1-y^2)V_{yy} - 2yV_y + \frac{\alpha^2}{4}V = 0, \tag{6.4.13}$$

where y and V are the new similarity variables given by,

$$y(\xi_1,\xi_2) = \sin\xi_1\sin\xi_2, \qquad V(y) = W. \tag{6.4.14}$$

Obviously, the solution of the Legendre equation, Eq.(6.4.13), is given by the following Legendre function:

$$V = c_1 LegendreP[0.5(\sqrt{\alpha^2+1}-1), y] + c_2 LegendreQ[0.5(\sqrt{\alpha^2+1}-1), y]. \tag{6.4.15}$$

Substituting for $y = \sin\theta\sin\phi$ and $v = u$ yields the following Legendre type solution:

$$\begin{aligned} u(t,\theta,\phi) = c_1 e^{-\frac{\alpha}{2}t} LegendreP[0.5(\sqrt{\alpha^2+1}-1), \sin\theta\sin\phi] \\ + c_2 e^{-\frac{\alpha}{2}t}.LegendreQ[0.5(\sqrt{\alpha^2+1}-1), \sin\theta\sin\phi]. \end{aligned} \tag{6.4.16}$$

Again, it is evident from the Legendre solution (6.4.16) that the amplitude of the wave vanishes as time evolves.

6.4.3 Reduction Under the Subalgebra $\chi_3 = \langle -\frac{2}{\alpha}X_1 + X_3, X_5\rangle$

The first reduction in this case is already performed in the previous case which led into Eq(6.4.11). Therefore, we deal only with second reduction via the rotational symmetry, X_5. It turns out that X_5 reduces Eq.(6.4.11) again into Legendre equation (6.4.13) but with different similarity variables given by,

$$y(\xi_1,\xi_2) = \sin\xi_1\cos\xi_2, \qquad V(y) = W. \tag{6.4.17}$$

Similarly, solving the Legendre equation (6.4.13) and transforming the similarity variables, ultimately leads to the following exact solution of Eq.(6.2.3):

$$\begin{aligned} u(t,\theta,\phi) = c_1 e^{-\frac{\alpha}{2}t} LegendreP[0.5(\sqrt{\alpha^2+1}-1), \sin\theta\cos\phi] \\ + c_2 e^{-\frac{\alpha}{2}t} LegendreQ[0.5(\sqrt{\alpha^2+1}-1), \sin\theta\cos\phi]. \end{aligned} \tag{6.4.18}$$

6.4.4 Reduction Under the Subalgebra $\chi_4 = \langle aX_1 + bX_2 + cX_3, dX_1 + eX_2 + fX_3\rangle$

In this case, the constants a, b, c, d, e and f are parameters that will be fixed later. The first level of reduction will be performed using the combination of symmetries, $X = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial\phi} + cu\frac{\partial}{\partial u}$. The characteristic system corresponding to this symmetry gives rise to the following similarity variables;

$$\xi_1(t,\theta,\phi,u)=\theta, \qquad \xi_2(t,\theta,\phi,u)=bt-a\phi, \qquad V(\xi_1,\xi_2,u)=ct-aln|u|.$$

Using the above variables, Eq(6.2.3) can be reduced into,

$$\begin{aligned}(a\csc^2\xi_1-\frac{b^2}{a})V_{\xi_2\xi_2}+(\frac{c}{a}-\frac{b}{a}V_{\xi_2})^2+\alpha(\frac{c}{a}-\frac{b}{a}V_{\xi_2})\\ +\frac{\cot\xi_1}{a}V_{\xi_1}+\frac{1}{a}V_{\xi_1\xi_1}-\frac{1}{a^2}V_{\xi_1}^2=\csc^2\xi_1V_{\xi_2}^2.\end{aligned} \tag{6.4.19}$$

In order to reduce Eq.(6.4.19), we consider the second level of reduction performed via the symmetry, $X=d\frac{\partial}{\partial t}+e\frac{\partial}{\partial\phi}+fu\frac{\partial}{\partial u}$. Consequently, Eq.(6.4.19) is reduced into,

$$\begin{aligned}(\frac{c}{a}+\frac{b(cd-af)}{a(ae-bd)})^2+\alpha(\frac{c}{a}+\frac{b(cd-af)}{a(ae-bd)})=-\frac{\cot x}{a(ae-bd)}W_x\\ -\frac{W_{xx}}{a(ae-bd)}+\frac{W_x^2}{a^2(ae-bd)^2}+\csc^2x\frac{(cd-af)^2}{(ae-bd)^2}\ .\end{aligned} \tag{6.4.20}$$

where x and W are the new similarity variables given by,

$$x(\xi_1,\xi_2)=\xi_1, \qquad W(x)=(cd-af)\xi_2+(ae-bd)V.$$

In order to further simplify Eq.(6.4.20), we fix the parameters as follows:

$$b=d=0, \quad a=e=f=1, \quad c=-\alpha.$$

Notice that our choice of the values of the parameters is valid since the considered subalgebra χ_4 in this case becomes $\chi_4=\langle\frac{\partial}{\partial t}-\alpha u\frac{\partial}{\partial u},\frac{\partial}{\partial\phi}+u\frac{\partial}{\partial u}\rangle$. It is clear that the commutation relation for this two dimensional subalgebra in this case is given by,

$$[\frac{\partial}{\partial t}-\alpha u\frac{\partial}{\partial u},\frac{\partial}{\partial\phi}+u\frac{\partial}{\partial u}]=0.$$

Hence, Substituting the chosen values of the parameters a, b, c, d, e and f in Eq.(6.4.20) yields,

$$\cot x W_x + W_{xx} - W_x^2 - \csc^2 x = 0. \tag{6.4.21}$$

We can solve Eq.(6.4.21) by making simplifying assumption. In particular, assume that,

$$W_x = \csc x h(x). \tag{6.4.22}$$

Then, substituting (6.4.22) in (6.4.21) gives,

$$h^2(x) - \sin x h'(x) + 1 = 0. \tag{6.4.23}$$

The solution of (6.4.23) is given by,

$$h(x) = \tan(c_1 + ln|\tan\frac{x}{2}|). \tag{6.4.24}$$

Putting Eq.(6.4.24) in Eq.(6.4.22) gives,

$$W_x = \csc x \tan(c_1 + ln|\tan\frac{x}{2}|). \tag{6.4.25}$$

Integrating Eq.(6.4.25) with respect to x yields,

$$W = -ln|\cos(c_1 + ln|\tan\frac{x}{2}|)| + c_2. \tag{6.4.26}$$

Thus, simple calculations show that the solution (6.4.26) is equivalent to the following exact solution of Eq.(6.2.3):

$$u(t, \theta, \phi) = Ke^{\phi - \alpha t}\cos(c_1 + ln|\tan\frac{\theta}{2}|). \tag{6.4.27}$$

A graph of the above solution for a fixed value of θ is shown below.

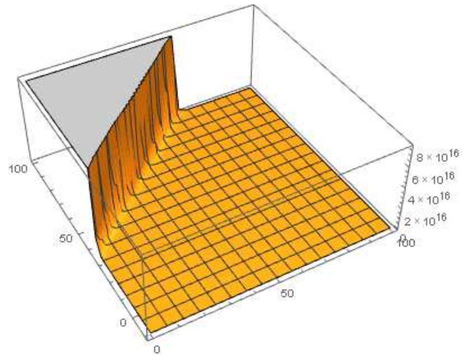

Fig. 6.2: A graph of solution (6.4.27) with fixed value of θ

A list of the solutions that have been obtained is shown in table 6.2.

Algebra	Exact solution
$\langle X_5, X_2 + X_4 \rangle$	$u = -\frac{K_1}{\alpha} e^{-\alpha t} + K_2$
$\langle -\frac{2}{\alpha} X_1 + X_3, X_4 \rangle$	$u = c_1 e^{-\frac{\alpha}{2} t} LegendreP[0.5(\sqrt{\alpha^2+1} - 1), \sin\theta \sin\phi]$ $+ c_2 e^{-\frac{\alpha}{2} t} LegendreQ[0.5(\sqrt{\alpha^2+1} - 1), \sin\theta \sin\phi]$
$\langle -\frac{2}{\alpha} X_1 + X_3, X_5 \rangle$	$u = c_1 e^{-\frac{\alpha}{2} t} LegendreP[0.5(\sqrt{\alpha^2+1} - 1), \sin\theta \cos\phi]$ $+ c_2 e^{-\frac{\alpha}{2} t} LegendreQ[0.5(\sqrt{\alpha^2+1} - 1), \sin\theta \cos\phi]$
$\langle X_1 - \alpha X_3, X_2 + X_3 \rangle$	$u = K e^{\phi - \alpha t} \cos(c_1 + ln\lvert \tan\frac{\theta}{2} \rvert)$

Tab. 6.2: Exact solutions of the damped wave equation

6.5 *Conclusion*

In this chapter, it has been shown that the damped wave equation on the surface of the sphere is spanned by five finite dimensional symmetry algebra which is the same algebra that span the classical wave equation on the sphere (see [53]). This means that coupling the wave equation with a damping term has no effect on the symmetry properties of the wave equation in the case of constant curvature of the sphere. Furthermore, four similarity reductions have been performed.

In each case of reduction, an exact solution has been obtained. The common feature of all obtained solutions is that the amplitude of the wave decreases as time evolves.

The study presented in this chapter paves the way for further investigation regarding the properties and solutions of the damped wave equations on non-flat background metric. In particular the damped wave equation on nonconstant curvature metric could be considered.

7. SOME SOLUTIONS & REDUCTIONS OF THE RELATIVISTIC HEAT EQUATION (RHE) IN FLAT FLRW SPACETIMES

In this chapter, we focus our attention on the temperature evolution of our universe. In order to address such issue, we study the relativistic heat equation on the flat FLRW spacetimes. We give a detailed introduction in the next section where we explain the significance of the Friedman metric and we present the derivation of the relativistic heat equation. The formulation of the problem is presented in section 7.2. In section 7.3, we implement the separation of variables method to obtain exact solutions of the studied heat equation. In section 7.4, we add proper boundary conditions to the considered problem to obtain explicit solutions that hold under certain simplifying assumption. A symmetry analysis of the relativistic heat equation is performed in section 7.5. These symmetries are utilized to perform some reductions of the RHE in section 7.6. Finally, discussions of the obtained results are provided in the last section.

7.1 ***Introduction***

The Friedmann spacetimes are extremely important in the study of the big bang cosmology which is the prevailing scientific model that describes the evolution of the universe [54]. The three Friedmann models, which are also called the Friedmann-Lematre-Robertson-Walker universe models, are described by a spacetime metric, given generally by,

$$ds^2 = dt^2 - a^2(t)[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + r^2 sin^2\theta)], \tag{7.1.1}$$

where the function $a(t)$ is called the scale factor [55]. The Eq.(7.1.1) represents three models of the universe, namely, open, flat and closed models for $k = -1$, $k = 0$ and $k = +1$ respectively (See figure 7.1).

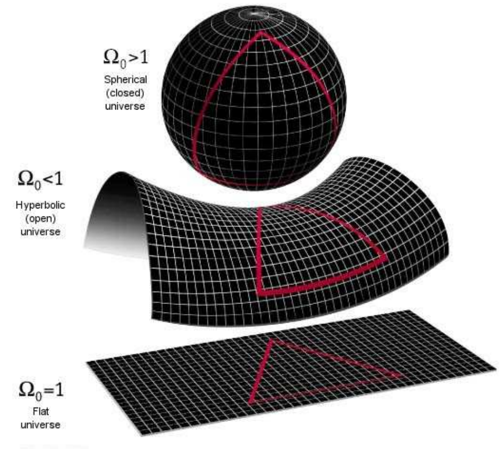

Fig. 7.1: The three models of the universe depending on the density parameter Ω_0

The construction of the Friedmann models is based on the observation that the universe in large scale is isotropic and homogenous. In fact, the standard model (7.1.1) was obtained by just following the geometric properties of homogeneity and isotropy and thus the Einstein field equations were not used to derive it. However, the Einstein equations are needed to determine the scale factor $a(t)$. According to calculations carried out by cosmologists, the scale factor function has different values depending on the evolution of the universe. In particular, the calculations have shown that for radiation-dominated era of the universe (which is believed to start after the inflation of the universe until 47000 years after the big bang), the value of $a(t)$ is given by, $a(t) \propto \sqrt{t}$. Then, after the radiation-dominated era, the universe passed through the matter-dominated era which took place in the period between 47,000 years and 9.8 billion years after the Big Bang. For this era, the calculations have shown that the scale factor is given by $a(t) \propto t^{\frac{2}{3}}$. Finally, the dark energy-dominated era which took place after the matter-dominated era corresponds to a scale factor given by $a(t) \propto e^{Ht}$, where H is the Hubble constant.

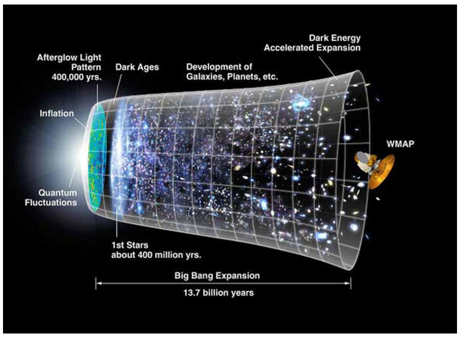

Fig. 7.2: Chronology of the universe according to the Big Bang theory

Since the purpose of this chapter is to present analytical solutions of the hyperbolic heat equation, we restrict ourselves to flat Friedman model only which is given in Cartesian coordinates as,

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2]. \tag{7.1.2}$$

The reader is referred to [6] for detailed explanation on the topics related to general relativity.

On the other hand, considering the fact that the linear wave equation in a background spacetime metric describes the propagation of energy and matter fields in the linearized regime, attempts have been made to solve wave equations in FLRW settings [56, 57]. The question one asks is how a relativistic heat equation can be dealt with? It is well known that the Fourier equation of heat conduction,

$$u_t = \alpha \triangle u, \tag{7.1.3}$$

in which $\alpha = k/(\rho c)$ representing thermal diffusivity, k thermal conductivity and $\triangle$ a Laplacian,

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \ , \tag{7.1.4}$$

is not compatible with the theory of Relativity because it assumes infinite speed of heat propagation [58,59]. Furthermore, it has been verified that, under certain circumstances, the Fourier equation violates the second law of thermodynamics as it allows heat to be transmitted from cold bodies to hot ones. To avoid this incompatibility, the hyperbolic heat conduction equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t} = \triangle u, \tag{7.1.5}$$

where $\beta > 0$ is the inverse of the diffusivity constant α, has been considered instead because it would allow a finite speed of heat propagation given by 'c'. This equation, Eq.(7.1.5), is similar to the telegraph equation for an electromagnetic field. In particular, this is a wave equation which allows for a range of phenomenon such as reflection, resonance and shock waves mostly uncommon to the diffusion process represented by Fourier law Eq.(7.1.3). Thus Eq.(7.1.5) is of more fundamental interest than Eq.(7.1.3). However, for this choice of using Eq.(7.1.5) instead of Eq.(7.1.3), it is required to modify the definition of the heat flux vector $\mathbf{q}$ from Fourier's linear model,

$$\mathbf{q} = -k\nabla u, \tag{7.1.6}$$

to

$$\tau_o \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k \triangle u, \tag{7.1.7}$$

whose detail can be found in [60,61], with the argument that Eq.(7.1.7) violates at least one statement of the law of thermodynamics [62–64].

The relativistic heat conduction model established the validity of Eq.(7.1.5) on the basis of Special Relativity theory alone [65]. However, in order to establish Eq.(7.1.5) to appeal to general relativistic setting correctly, it needs to be re-cast in the form,

$$\beta \frac{\partial u}{\partial t} = \Box u, \tag{7.1.8}$$

with modified heat flux vector $\mathbf{q} = -k\Box u$, where $\Box$ operator embodies coupling with the 4- spacetime geometry via,

$$\Box = \frac{1}{\sqrt{\mid g \mid}} \sum_{i=0}^{3} \frac{\partial}{\partial x_j} (\sqrt{\mid g \mid} g^{ij} \frac{\partial}{\partial x_j}). \tag{7.1.9}$$

Even though, the above situation settles most of the issues to deal with relativistic hyperbolic equation, a question that still needs justification is, how one shall couple the left hand side of Eq.(7.1.8) in the spacetime settings. To incorporate this point, it is argued that the only model where problem can be meaningfully resolved are the three dynamical Friedman universe models, where the $g_{oo} = 1$; and therefore do not pose issues in solving the weakly defined hyperbolic heat equation.

7.2 ***Problem Formulation***

Applying Eq.(7.1.9) to the spacetime metric (7.1.2) gives rise to the following equation:

$$\Box u = \frac{3\dot{a}(t)}{a(t)} u_t + u_{tt} - \frac{u_{xx}}{a^2(t)} - \frac{u_{yy}}{a^2(t)} - \frac{u_{zz}}{a^2(t)}. \tag{7.2.1}$$

Therefore, by substituting Eq.(7.2.1) in Eq.(7.1.8), we derive the RHE on the flat Friedman spacetime as,

$$[\beta a^2(t) - 3a(t)\dot{a}(t)]u_t - a^2(t)u_{tt} + u_{xx} + u_{yy} + u_{zz} = 0. \tag{7.2.2}$$

In this work, we intend to implement the method of separation of variables to obtain solutions of Eq.(7.2.2). The obtained solutions correspond to radiation-dominated, matter-dominated and dark energy-dominated universes. Furthermore, the symmetry method is used to reduce the considered equation to a second order PDE of two independent variables.

7.3 *Solutions of the RHE*

In order to solve the RHE, we focus on the method of separation of variables. To implement this method, we separate the time variable t from the space variables (x, y, z) by assuming a solution of the form,

$$u(t, x, y, z) = T(t)S(x, y, z). \tag{7.3.1}$$

So Eq.(7.2.2) becomes,

$$[\beta a^2(t) - 3a(t)a'(t)]T'(t)S - a^2(t)T''(t)S + T(t)S_{xx} + T(t)S_{yy} + T(t)S_{zz} = 0.$$

We recast the above equation by $T(t)S$ to obtain,

$$[\beta a^2(t) - 3a(t)a'(t)]\frac{T'(t)}{T(t)} - a^2(t)\frac{T''(t)}{T(t)} + \frac{S_{xx}}{S} + \frac{S_{yy}}{S} + \frac{S_{zz}}{S} = 0. \tag{7.3.2}$$

Equivalently, Eq.(7.3.2) can be written as,

$$[\beta a^2(t) - 3a(t)a'(t)]\frac{T'(t)}{T(t)} - a^2(t)\frac{T''(t)}{T(t)} = -\frac{S_{xx}}{S} - \frac{S_{yy}}{S} - \frac{S_{zz}}{S} = \alpha, \quad (7.3.3)$$

where α is a constant that can be zero, positive or negative. We consider each case in details as follows.

Case I $(\alpha = 0)$

In this case, Eq.(7.3.3) can be written as two separate equations as follows:

$$S_{xx} + S_{yy} + S_{zz} = 0, \quad (7.3.4)$$

$$[\beta a^2(t) - 3a(t)a'(t)]T'(t) - a^2(t)T''(t) = 0. \quad (7.3.5)$$

Notice that Eq.(7.3.4) can be again solved using separation of variables by assuming that $S = X(x)Y(y)Z(z)$, which enables one to write Eq.(7.3.4) as,

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0.$$

Recasting the above equation we get,

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0.$$

Equivalently, we can write the above equation as,

$$\frac{Z''(z)}{Z(z)} = -\frac{X''(x)}{X(x)} - \frac{Y''(y)}{Y(y)} = \lambda^2. \quad (7.3.6)$$

Thus, we have the following equation:

$$-\frac{X''(x)}{X(x)} = \lambda^2 + \frac{Y''(y)}{Y(y)} = k^2. \quad (7.3.7)$$

Therefore, from Eq.(7.3.6) and Eq.(7.3.7), we derive the following three equations:

$$X''(x) + k^2 X(x) = 0, \tag{7.3.8}$$

$$Y''(y) + (\lambda^2 - k^2)Y(y) = 0, \tag{7.3.9}$$

$$Z''(z) - \lambda^2 Z(z) = 0. \tag{7.3.10}$$

The solutions of equations (7.3.8), (7.3.9) and (7.3.10) are given respectively as,

$$X = c_1 coskx + c_2 sinkx,$$

$$Y = c_3 cos\sqrt{\lambda^2 - k^2}y + c_4 sin\sqrt{\lambda^2 - k^2}y,$$

$$Z = c_5 cosh\lambda z + c_6 sinh\lambda z.$$

Consequently, the solution of Eq.(7.3.4) is given as,

$$S = (c_1 coskx + c_2 sinkx)(c_3 cos\sqrt{\lambda^2 - k^2}y + c_4 sin\sqrt{\lambda^2 - k^2}y)(c_5 cosh\lambda z + c_6 sinh\lambda z). \tag{7.3.11}$$

It is straightforward to conclude that the solution of Eq.(7.3.5) is given by,

$$T(t) = \int \frac{e^{\beta t}}{a^3(t)} dt. \tag{7.3.12}$$

Therefore, the solution of the RHE is obtained by putting equations (7.3.11) and (7.3.12) in Eq.(7.3.1) to get,

$$\begin{aligned} u = &(c_1 coskx + c_2 sinkx)(c_3 cos\sqrt{\lambda^2 2 - k^2}y + c_4 sin\sqrt{\lambda^2 - k^2}y)\times \\ &(c_5 cosh\lambda z + c_6 sinh\lambda z)\int \frac{e^{\beta t}}{a^3(t)} dt. \end{aligned} \tag{7.3.13}$$

We can apply the above obtained solution (7.3.13) for specific interesting values of the scale factor $a(t)$.

Recall that for radiation-dominated universe we have $a(t) = \sqrt{t}$ so the above solution becomes,

$$u = (c_1 coskx + c_2 sinkx)(c_3 cos\sqrt{\lambda^2 - k^2}y + c_4 sin\sqrt{\lambda^2 - k^2}y)\times$$
$$(c_5 cosh\lambda z + c_6 sinh\lambda z)\int \frac{e^{\beta t}}{t^{\frac{3}{2}}}dt.$$

We can expand the exponential function as a Taylor series in above solution so we can re-write it as,

$$u = (c_1 coskx + c_2 sinkx)(c_3 cos\sqrt{\lambda^2 - k^2}y + c_4 sin\sqrt{\lambda^2 - k^2}y)\times$$
$$(c_5 cosh\lambda z + c_6 sinh\lambda z)\int \sum_{n=0}^{\infty} \frac{(\beta t)^n}{n! t^{\frac{3}{2}}}dt.$$

Therefore, the solution for **radiation-dominated universe** is given by,

$$u_{rad} = (c_1 coskx + c_2 sinkx)(c_3 cos\sqrt{\lambda^2 - k^2}y + c_4 sin\sqrt{\lambda^2 - k^2}y)\times$$
$$(c_5 cosh\lambda z + c_6 sinh\lambda z)\sum_{n=0}^{\infty} \frac{\beta^n t^{n-\frac{1}{2}}}{(n - \frac{1}{2})\Gamma(n+1)} \ . \tag{7.3.14}$$

Similarly, for matter-dominated universe, we have $a(t) = t^{\frac{2}{3}}$ so the solution (7.3.13) becomes,

$$u = (c_1 coskx + c_2 sinkx)(c_3 cos\sqrt{\lambda^2 - k^2}y + c_4 sin\sqrt{\lambda^2 - k^2}y)\times$$
$$(c_5 cosh\lambda z + c_6 sinh\lambda z)\int \frac{e^{\beta t}}{t^2}dt.$$

Again, we expand the exponential function inside the integral to get,

$$u = (c_1 coskx + c_2 sinkx)(c_3 cos\sqrt{\lambda^2 - k^2}y + c_4 sin\sqrt{\lambda^2 - k^2}y)\times$$
$$(c_5 cosh\lambda z + c_6 sinh\lambda z)\int \sum_{n=0}^{\infty} \frac{(\beta t)^n}{n! t^2}dt.$$

Therefore, the solution for **matter-dominated universe** is given by,

$$\begin{aligned} u_{mat} = & (c_1 coskx + c_2 sinkx)(c_3 cos\sqrt{\lambda^2 - k^2}y + c_4 sin\sqrt{\lambda^2 - k^2}y)\times \\ & (c_5 cosh\lambda z + c_6 sinh\lambda z)\big(\beta lnt - \frac{1}{t} + \sum_{n=2}^{\infty} \frac{\beta^n}{(n-1)\Gamma(n+1)} t^{n-1}\big). \end{aligned} \tag{7.3.15}$$

Finally, for dark energy-dominated universe we have $a(t) = e^{Ht}$ so the solution in this case becomes,

$$\begin{aligned} u_{dark} = & (c_1 coskx + c_2 sinkx)(c_3 cos\sqrt{\lambda^2 - k^2}y + c_4 sin\sqrt{\lambda^2 - k^2}y)\times \\ & (c_5 cosh\lambda z + c_6 sinh\lambda z)(\frac{c_7}{\beta - 3H} e^{(\beta - 3H)t} + c_8). \end{aligned} \tag{7.3.16}$$

Notice that the solution obtained for dark energy-dominated universe is a closed form solution and we can impose physical restrictions on this solution to make it more consistent with our actual universe.

If we assume the universe keeps expanding and it does not suffer a Big Crunch, then photons of radiation are stretched in wavelength, giving rise to the cosmological redshift. This causes a decrease in the intensity of the light emitted and indicates a lower temperature. Therefore, at infinite time, the temperature of the microwave background radiation will be close to absolute zero.

Mathematically speaking, $u \to 0$ as $t \to \infty$. Applying this condition to the solution (7.3.16) we get $c_8 = 0$ and $\beta < 3H$. Thus , the solution for **dark energy-dominated universe** is reduced to,

$$\begin{aligned} u_{dark} = & \frac{c_7}{\beta - 3H}(c_1 coskx + c_2 sinkx)(c_3 cos\sqrt{\lambda^2 - k^2}y + c_4 sin\sqrt{\lambda^2 - k^2}y)\times \\ & (c_5 cosh\lambda z + c_6 sinh\lambda z)e^{(\beta - 3H)t}. \end{aligned} \tag{7.3.17}$$

Case II ($\alpha > 0$)

We restrict our attention to positive value of α, i.e we assume $\alpha = \lambda^2$. Following the procedure as above, in this case, Eq.(7.3.3) gives rise to the following two equations:

$$S_{xx} + S_{yy} + S_{zz} = -\lambda^2 s, \tag{7.3.18}$$

$$[\beta a^2(t) - 3a(t)a'(t)]T'(t) - a^2(t)T''(t) - \lambda^2 T(t) = 0. \tag{7.3.19}$$

Eq.(7.3.18) is a Poisson-type equation that can be easily solved using the separation of variables method again. By assuming a solution of the form $S = X(x)Y(y)Z(z)$ and following the procedure outlined in the first case for Laplace equation, one arrives at the following solution of Eq.(7.3.18):

$$\begin{aligned} S = (c_1 \cos\sqrt{\lambda^2 - k^2}\ x + c_2 sin\sqrt{\lambda^2 - k^2}\ x)(c_3 \cos mz + c_4 \sin mz)\times \\ (c_5 \cos\sqrt{k^2 - m^2}\ y + c_6 \sin\sqrt{k^2 - m^2}\ y). \end{aligned} \tag{7.3.20}$$

where λ, m and k are constants satisfying $m < k < \lambda$. To solve Eq.(7.3.19), we restrict ourselves to the three values of the scale factor as discussed earlier. For radiation dominated universe, Eq.(7.3.19) takes the following form:

$$[\beta\ t - \frac{3}{2}]T'(t) - tT''(t) - \lambda^2 T(t) = 0. \tag{7.3.21}$$

Solving above equation using Mathematica yields,

$$\begin{aligned} T(t) = c_7 e^{\frac{1}{2}(\beta - \sqrt{\beta^2 - 4\lambda^2})t} Hypergeom\ U[\frac{3}{4} - \frac{3\beta}{\sqrt{\beta^2 - 4\lambda^2}}, \frac{3}{2}, \sqrt{\beta^2 - 4\lambda^2}t] + \\ c_8 e^{\frac{1}{2}(\beta - \sqrt{\beta^2 - 4\lambda^2})t} LaguerreL[\frac{\beta}{\sqrt{\beta^2 - 4\lambda^2}} - 1, \frac{1}{2}, \sqrt{\beta^2 - 4\lambda^2}t]. \end{aligned} \tag{7.3.22}$$

Putting equations (7.3.20) and (7.3.22) in Eq.(7.3.1) yields the following solution for **radiation dominated universe:**

$u_{rad} = [(c_1 \cos\sqrt{\lambda^2 - k^2}\ x + c_2 sin\sqrt{\lambda^2 - k^2}\ x)(c_3 \cos mz + c_4 \sin mz)(c_5 \cos\sqrt{k^2 - m^2}\ y + c_6 \sin\sqrt{k^2 - m^2}\ y)][c_7 e^{\frac{1}{2}(\beta - \sqrt{\beta^2 - 4\lambda^2})t} Hypergeom\ U[\frac{3}{4} - \frac{3\beta}{\sqrt{\beta^2 - 4\lambda^2}}, \frac{3}{2}, \sqrt{\beta^2 - 4\lambda^2}t] +$

$c_8e^{\frac{1}{2}(\beta-\sqrt{\beta^2-4\lambda^2})t}LaguerreL[\frac{\beta}{\sqrt{\beta^2-4\lambda^2}}-1,\frac{1}{2},\sqrt{\beta^2-4\lambda^2}t]]$.

Similarly, for matter dominated universe, Eq.(7.3.19) becomes,

$$T''(t)+(\frac{2}{t}-\beta)T'(t)-\lambda^2T(t)=0. \tag{7.3.23}$$

Solving above equation using Mathematica yields

$$\begin{aligned}T(t)=c_7e^{\frac{1}{2}(\beta-\sqrt{\beta^2+4\lambda^2})t}Hypergeom\ 1F1[1-\frac{\beta}{\sqrt{\beta^2+4\lambda^2}},2,\sqrt{\beta^2+4\lambda^2}t]+\\ c_8e^{\frac{1}{2}(\beta-\sqrt{\beta^2+4\lambda^2})t}Hypergeom\ U[1-\frac{\beta}{\sqrt{\beta^2+4\lambda^2}},2,\sqrt{\beta^2+4\lambda^2}t].\end{aligned} \tag{7.3.24}$$

Thus, putting equations (7.3.20) and (7.3.24) in Eq.(7.3.1) yields the following solution for **matter dominated universe:**

$u_{mat}=[(c_1\cos\sqrt{\lambda^2-k^2}\ x+c_2sin\sqrt{\lambda^2-k^2}\ x)(c_3\cos mz+c_4\sin mz)(c_5\cos\sqrt{k^2-m^2}\ y+c_6\sin\sqrt{k^2-m^2}\ y)][c_7e^{\frac{1}{2}(\beta-\sqrt{\beta^2+4\lambda^2})t}Hypergeom\ 1F1[1-\frac{\beta}{\sqrt{\beta^2+4\lambda^2}},2,\sqrt{\beta^2+4\lambda^2}t]+c_8e^{\frac{1}{2}(\beta-\sqrt{\beta^2+4\lambda^2})t}Hypergeom\ U[1-\frac{\beta}{\sqrt{\beta^2+4\lambda^2}},2,\sqrt{\beta^2+4\lambda^2}t]]$.

Finally, for dark energy dominated universe, Eq.(7.3.19) becomes,

$$T''(t)+(3H-\beta)T'(t)-\lambda^2e^{-2Ht}T(t)=0, \tag{7.3.25}$$

and solving using Mathematica yields the following solution:

$T(t)=c_7\sqrt{(2H)^{\frac{\beta}{H}-3}}e^{-2Ht(\frac{3}{4}-\frac{\beta}{4H})}(\frac{k}{H})^{\frac{3}{2}-\frac{\beta}{H}}Bessel\ J[\frac{3}{2}-\frac{\beta}{2H},\frac{ke^{-Ht}}{H}]Gamma[\frac{5}{2}-\frac{\beta}{2H}]+c_8\sqrt{(2H)^{\frac{\beta}{H}-3}}e^{-2Ht(\frac{3}{4}-\frac{\beta}{4H})}Bessel\ J[\frac{\beta-3H}{2H},\frac{ke^{-Ht}}{H}]Gamma[\frac{\beta}{2H}-\frac{1}{2}]$.

Therefore, putting above equation and Eq.(7.3.20) in Eq.(7.3.1) yields the following solution for **dark energy dominated universe:**

$u_{dark}=[(c_1\cos\sqrt{\lambda^2-k^2}\ x+c_2sin\sqrt{\lambda^2-k^2}\ x)(c_3\cos mz+c_4\sin mz)(c_5\cos\sqrt{k^2-m^2}\ y+c_6\sin\sqrt{k^2-m^2}\ y)][c_7\sqrt{(2H)^{\frac{\beta}{H}-3}}e^{-2Ht(\frac{3}{4}-\frac{\beta}{4H})}(\frac{k}{H})^{\frac{3}{2}-\frac{\beta}{H}}Bessel\ J[\frac{3}{2}-\frac{\beta}{2H},\frac{ke^{-Ht}}{H}]Gamma[\frac{5}{2}-\frac{\beta}{2H}]+c_8\sqrt{(2H)^{\frac{\beta}{H}-3}}e^{-2Ht(\frac{3}{4}-\frac{\beta}{4H})}Bessel\ J[\frac{\beta-3H}{2H},\frac{ke^{-Ht}}{H}]Gamma[\frac{\beta}{2H}-\frac{1}{2}]]$.

Case III $\quad(\alpha<0)$

In this case, Eq.(7.3.3) gives rise to the following two equations:

$$S_{xx}+S_{yy}+S_{zz}=\lambda^2s, \tag{7.3.26}$$

$$[\beta a^2(t) - 3a(t)a'(t)]T'(t) - a^2(t)T''(t) + \lambda^2 T(t) = 0. \tag{7.3.27}$$

In fact, using mathematica one can show that the equations (7.3.26) and (7.3.27) have complex valued solutions for the considered universe models. Therefore, this case will be ignored as it does not lead to any physically meaningful solutions.

7.4 *Solution Under Boundary Conditions*

We restrict our attention in this section to the dark energy dominated universe since it gives rise to a closed form solution. Since the variation of temperature of the microwave background radiation at different positions of space is small, one can neglect these variations and consider all points of space to be identical. Thus, if we let $u_x = u_y = u_z \approx 0$, then the RHE (7.2.2) is reduced to the following equation:

$$[\beta a^2(t) - 3a(t)\dot{a}(t)]u_t - a^2(t)u_{tt} \approx 0, \tag{7.4.1}$$

giving rise to the following solution:

$$u_{dark} \approx u_0 e^{(\beta - 3H)t}. \tag{7.4.2}$$

Notice that for sufficiently small β, the above solution (7.4.2) suggests that the temperature of the microwave background radiation is proportional to the inverse of the cube of the scale factor in agreement with the result obtained in standard cosmology. Furthermore, we could add proper boundary conditions by considering the data provided by recent measurement which indicates that the temperature of the microwave background radiation is 2.7 Kelvin. Also, taking into account that the temperature in the beginning of the dark energy dominated era was approximately 4 Kelvin [66], then one may determine the constants u_0 and β by considering the boundary conditions $u(0) = 4$ and $u(4) = 2.7$, where we have taken the time unit to be Ga ($Ga = 10^9$ years), the temperature in Kelvin and the age of the universe is assumed to be 14 billion years. A straightforward calculation shows that the solution (7.4.2) under the mentioned boundary conditions can be written as,

$$u_{dark} \approx 4(\frac{27}{40})^{\frac{t}{4}}. \tag{7.4.3}$$

Obviously, the above solution suggests that the temperature of the universe is decreasing by a scale of exponential function as shown in figure 7.3 below.

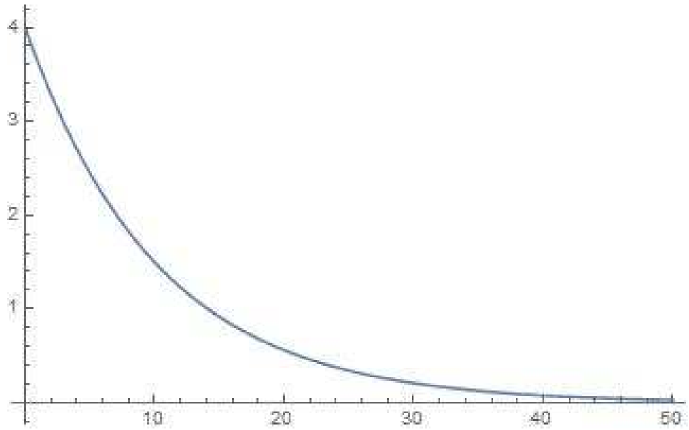

Fig. 7.3: Temperature evolution of the universe according to solution (7.4.3).

7.5 ***The Symmetry Algebra***

Another approach to address our problem is to implement the Lie symmetry method. As we have seen in previous chapter, This method enables one to reduce the number of the independent variables in the considered equation. In particular, we can reduce the considered RHE to a second order PDE of two independent variables by utilizing its symmetries.

Notice that the symmetry generator corresponding to Eq.(7.2.2) is given by,

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial z} + \phi\frac{\partial}{\partial u}, \tag{7.5.1}$$

where the coefficients τ, ξ, η, γ and ϕ are functions of t, x, y, z and u. Since Eq.(7.2.2) involves partial derivatives of u, we need to prolong the symmetry generator (7.5.1) to the second order so that it involves the transformations of these partial derivative as well. This prolongation is expressed as a new vector field given as,

$$X^[2] = X + \psi^t \frac{\partial}{\partial u_t} + \psi^{tt} \frac{\partial}{\partial u_{tt}} + \psi^{xx} \frac{\partial}{\partial u_{xx}} + \psi^{yy} \frac{\partial}{\partial u_{yy}} + \psi^{zz} \frac{\partial}{\partial u_{zz}} \ . \tag{7.5.2}$$

The invariance criterion for obtaining the Lie symmetries of Eq.(7.2.2) is given by,

$$X^2 F|_{F=0} = 0, \tag{7.5.3}$$

where $F = [\beta a^2(t) - 3a(t)\dot{a}(t)]u_t - a^2(t)u_{tt} + u_{xx} + u_{yy} + u_{zz}$. The criterion (7.5.3) leads to the following system of differential equations:

$$\tau_u = \xi_u = \eta_u = \gamma_u = \phi_{uu} = 0, \tag{7.5.4}$$

$$\xi_z + \gamma_x = 0, \tag{7.5.5}$$

$$\xi_y + \eta_x = 0, \tag{7.5.6}$$

$$\eta_z + \gamma_y = 0, \tag{7.5.7}$$

$$\eta_y - \xi_x = 0, \tag{7.5.8}$$

$$\gamma_z - \xi_x = 0, \tag{7.5.9}$$

$$a^2(t)\xi_t - \tau_x = 0, \tag{7.5.10}$$

$$a^2(t)\eta_t - \tau_y = 0, \tag{7.5.11}$$

$$a^2(t)\gamma_t - \tau_z = 0, \tag{7.5.12}$$

$$\tau a'(t) - a(t)\tau_t + a(t)\xi_x = 0, \tag{7.5.13}$$

$$a^2(t)[(3-\beta)\phi_t + \phi_{tt}] - \phi_{xx} - \phi_{yy} - \phi_{zz} = 0, \tag{7.5.14}$$

$$\begin{aligned} &a^2(t)[(3-\beta)\tau_t - 2\phi_{tu} + \tau_{tt} + (2\beta - 6)\xi_x] \\ &+2(\beta-3)a(t)a'(t)\tau - \tau_{xx} - \tau_{yy} - \tau_{zz} = 0, \end{aligned} \tag{7.5.15}$$

$$a^2(t)[(\beta-3)\xi_t - \xi_{tt}] - 2\phi_{xu} + \xi_{xx} + \xi_{yy} + \xi_{zz} = 0, \tag{7.5.16}$$

$$a^2(t)[(\beta-3)\eta_t - \eta_{tt}] - 2\phi_{yu} + \eta_{xx} + \eta_{yy} + \eta_{zz} = 0, \qquad (7.5.17)$$

$$a^2(t)[(\beta-3)\gamma_t - \gamma_{tt}] - 2\phi_{zu} + \gamma_{xx} + \gamma_{yy} + \gamma_{zz} = 0. \qquad (7.5.18)$$

Solving the above determining system gives rise to the following infinitesimals:

$$\tau = 0, \qquad \xi = c_1 + c_5 y - c_6 z, \qquad \eta = c_2 - c_5 x + c_7 z,$$

$$\gamma = c_3 + c_6 x - c_7 y, \qquad \phi = c_4 u + A(t,x,y,z),$$

where $A(t,x,y,z)$ is any function satisfying Eq.(7.2.2). The seven infinitesimal symmetry generators associated with the above infinitesimals are given by,

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial z}, \qquad X_4 = u\frac{\partial}{\partial u},$$

$$X_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad X_6 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \qquad X_7 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$

$$X_\infty = A(t,x,y,z)\frac{\partial}{\partial u}\ .$$

At this stage, we construct the commutator table for the derived symmetries as shown in table 7.1 below.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	0	0	0	$-X_2$	X_3	0
X_2	0	0	0	0	X_1	0	$-X_3$
X_3	0	0	0	0	0	$-X_1$	X_2
X_4	0	0	0	0	0	0	0
X_5	X_2	$-X_1$	0	0	0	$-X_7$	X_6
X_6	$-X_3$	0	X_1	0	X_7	0	$-X_5$
X_7	0	X_3	$-X_2$	0	$-X_6$	X_5	0

Tab. 7.1: Commutator table for the symmetries of the RHE (7.2.2)

‘

7.6 Reductions of the RHE

In order to reduce the RHE (7.2.2) by two variables, we need to use two dimensional subalgebras. In particular, for this research we restrict our attention on two subalgebras given by,

$$\chi_1 = \langle X_1 + X_2, X_2 + X_3 \rangle, \qquad \chi_2 = \langle X_5, X_3 + X_4 \rangle.$$

7.6.1 Reductions under $\chi_1 = \langle X_1 + X_2, X_2 + X_3 \rangle$

The first level of reduction can be performed using the symmetry $X = X_1 + X_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. This symmetry gives rise to the following characteristic system:

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{1} = \frac{dz}{0} = \frac{du}{0}. \tag{7.6.1}$$

Integrating Eq.(7.6.1) leads to the following similarity variables:

$$\xi_1(t, x, y, z, u) = y - x, \tag{7.6.2}$$

$$\xi_2(t, x, y, z, u) = z, \tag{7.6.3}$$

$$\xi_3(t, x, y, z, u) = t, \tag{7.6.4}$$

$$v(\xi_1, \xi_2, \xi_3) = u. \tag{7.6.5}$$

Notice that the similarity variables are invariant functions that have been obtained from the constants of integration of the characteristic system (7.6.1). Now, substituting the similarity variables in Eq.(7.2.2) gives,

$$[\beta a^2(\xi_3) - 3a(\xi_3)a'(\xi_3)]v_{\xi_3} - a^2(\xi_3)v_{\xi_3\xi_3} + 2v_{\xi_1\xi_1} + v_{\xi_2\xi_2} = 0. \tag{7.6.6}$$

The second level of reduction is performed using the symmetry $X_2 + X_3 = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Notice that Eq.(7.6.6) inherits the symmetry,

$$X(\xi_1)\frac{\partial}{\partial \xi_1} + X(\xi_2)\frac{\partial}{\partial \xi_2} + X(\xi_3)\frac{\partial}{\partial \xi_3} + X(v)\frac{\partial}{\partial v} = \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2}.$$

The above inherited symmetry corresponds to the following characteristic system:

$$\frac{d\xi_1}{1} = \frac{d\xi_2}{1} = \frac{d\xi_3}{0} = \frac{dv}{0}. \tag{7.6.7}$$

Solving the characteristic system (7.6.7) leads to new similarity variables given as follow,

$$s_1(\xi_1, \xi_2, \xi_3) = \xi_1 - \xi_2, \tag{7.6.8}$$

$$s_2(\xi_1, \xi_2, \xi_3) = \xi_3, \tag{7.6.9}$$

$$w(s_1, s_2) = v. \tag{7.6.10}$$

Substituting the new similarity variables in Eq.(7.6.6) reduces it to the following equation:

$$[\beta a^2(s_2) - 3a(s_2)a'(s_2)]w_{s_2} - a^2(s_2)w_{s_2 s_2} + 3w_{s_1 s_1} = 0. \tag{7.6.11}$$

Thus, the RHE (7.2.2) has been reduced into Eq.(7.6.11) that involves two independent variables which means that two variables have been eliminated.

7.6.2 **Reductions under** $\chi_2 = \langle X_5, X_3 + X_4 \rangle$

In this subsection we follow the same procedure as in the previous subsection to obtain another reduction of the RHE (7.2.2).
The first level of reduction is performed using $X_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. This symmetry reduces Eq.(7.2.2) into the following equation:

$$[\beta a^2(\xi_1) - 3a(\xi_1)a'(\xi_1)]v_{\xi_1} - a^2(\xi_1)v_{\xi_1\xi_1} + 4v_{\xi_3} + 4\xi_3 v_{\xi_3\xi_3} + v_{\xi_2\xi_2} = 0, \tag{7.6.12}$$

where the similarity variables are given by:

$$\xi_1(t, x, y, z, u) = t, \tag{7.6.13}$$

$$\xi_2(t, x, y, z, u) = z, \tag{7.6.14}$$

$$\xi_3(t, x, y, z, u) = x^2 + y^2, \tag{7.6.15}$$

$$v(\xi_1, \xi_2, \xi_3) = u. \tag{7.6.16}$$

The second level of reduction is performed using the symmetry $X_3 + X_4 = \frac{\partial}{\partial z} + u\frac{\partial}{\partial u}$. This symmetry ultimately reduces Eq.(7.6.12) to the following equation:

$$\begin{aligned}[\beta a^2(s_1) - 3a(s_1)a'(s_1)]w_{s_1} - a^2(s_1)w_{s_1 s_1} - a^2(s_1)w_{s_1}^2 \\ +4w_{s_2} + 4s_2 w_{s_2 s_2} + 4s_2 w_{s_2}^2 + 1 = 0,\end{aligned} \tag{7.6.17}$$

where the new similarity variables are given by:

$$s_1(\xi_1, \xi_2, \xi_3) = \xi_1, \tag{7.6.18}$$

$$s_2(\xi_1, \xi_2, \xi_3) = \xi_3, \tag{7.6.19}$$

$$w(s_1, s_2) = \log v - \xi_2. \tag{7.6.20}$$

Notice that one can in principle obtain new exact solutions to the RHE by further investigating the reduced equations (7.6.11) and (7.6.17). However, it is beyond the scope of this chapter to investigate these equations further since it is hard to obtain closed form solutions in which physical conclusions could be drawn.

7.7 *Conclusion*

In this chapter, we have implemented the separation of variables method to obtain exact solutions of the relativistic heat equation in flat Friedmann spacetime. The obtained solutions correspond to radiation-dominated , matter-dominated and dark energy-dominated universes. In particular, for dark energy-dominated universe, we were able to obtain a closed form solution given by Eq.(7.3.17) which, under further investigation, enabled us to recover some significant results obtained in standard cosmology.

Furthermore, using a simplifying assumption and proper boundary conditions we could reduce solution (7.3.17) into the solution (7.4.3) which indicates that the temperature of the universe drops as it expands by a scale of exponential function. This means that the temperature of the cosmic microwave background radiation is decreasing very fast on cosmological scale. For example, after 60 billion years ($t = 64\ Ga$), calculations based on solution (7.4.3) indicates that the temperature of the microwave background radiations will be as small as 0.007 kelvin, which make them undetectable in the far future.

Furthermore, applying the symmetry method to the RHE (7.2.2) reveals that the equation admits seven finite dimensional symmetry algebra. The time like Killing vector $\frac{\partial}{\partial t}$ was not admitted because the scale factor $a(t)$ depends explicitly on t which makes the considered metric non-static. The derived symmetries have been utilized to perform two similarity reductions. Although the reduced equations have not been solved in this research, one may obtain new exact solutions of the RHE by further investigating these reduced equations.

8. CONCLUSION AND FUTURE WORK

This dissertation evolves around discussion of evolution type equations and conservation laws obeyed by physical systems. As far as evolution type equations are concerned, they are extremely important in modeling many interesting problems in science and engineering. In particular, damped wave equations are of interest as they represent a very powerful tool in formulating many real life problems. We are frequently led to such equations when dealing with physical systems that dissipate energy as we have mentioned earlier. As far as the pseudo-Riemannian geometries are concerned, there is a hunch that these equations may have wider applications in explaining problems related to energy transport in cosmological context. However, it is not clear a priori how such equations can be modeled in such geometries generally. This dissertation is devoted to a study of these questions. In order to provide answers to understand the invariant properties possessed by the physical systems, it is also essential to study conservation laws admitted by the physical systems. Apart from their being providing interesting physical interpretations, the conservation laws are also important in providing the integrability of the systems. We have focused a part of the dissertation to this aspect also.

The most general attempt to investigate the variational symmetries for plane symmetric non-static spacetimes has been performed in chapter 3. This attempt led us to the derivation of new results from Noether symmetries for the considered spacetime metrics. These results are applied to some interesting spacetime metrics in General Relativity, recovering results already available in literature.

The damped wave equations in one and higher dimensions have been studied in chapters 4 and 5, respectively, where symmetry classifications have been performed for each equation. In both cases, we have provided all possible forms of the studied equation for which larger symmetry algebra exists. These symmetries have also been utilized to obtain exact solutions of the considered equations.

Furthermore, the investigation of damped wave equations has been extended to an interesting metric representing the surface of a sphere in chapter 6. The symmetry analysis of the equation indicated that the damping term has no effect on its symmetry structure. This fact has been verified by comparing the Lie symmetries of the classical wave equation on the sphere with its corresponding damped equation. Moreover, symmetries of the damped wave equation on the sphere have been implemented to derive new exact solutions. All obtained solutions describe, as predicted, waves whose amplitude of oscillation decreases as the time evolves.

Keeping in mind the significance of investigating the thermal history of our actual universe, a successful attempt has been made , in chapter 7, to study the relativistic heat equation in flat Friedmann spacetime. By implementing the separation of variables method, new exact solutions, corresponding to radiation-dominated, matter-dominated and dark energy-dominated universes, have been found. In particular, the solution of dark energy-dominated universe suggests that the temperature of the cosmic microwave background radiation decreases by a scale of exponential function. Our investigation of the relativistic heat equation revealed that some significant results obtained in standard cosmology could be recovered through the solutions we have obtained. It is interesting to note that some of the results we have obtained were similar to the cosmologists' results despite the fact that their approach to tackle this problem was totally different from our approach.

The research conducted in this dissertation opens doors for new directions for future work. In particular, it may be a worthwhile approach to see how a diffusion and damped equations can be modeled and what understanding they give for the more general spacetime, not considered in this dissertation.

REFERENCES

[1] J M Lee: Introduction to Smooth Manifolds. Springer, second edition (2012).

[2] Peter Petersen: Riemannian Geometry. Berlin, Springer-Verlag,ISBN 0-387-98212-4 (2006).

[3] B O Neill, Semi-Riemannian geometry with applications to Relativity. Academic Press, New York, 1983.

[4] Jose Natario: General Relativity without Calculus, A concise Introduction to the geometry of relativity. Springer (2011).

[5] Jerry B. Griffiths and Jiri Podolsky: Exact Spacetimes in Einstein's General Relativity. Cambridge University (2009).

[6] C W Misner, K S Thorne and J A Wheeler: Gravitation. W H Freeman (1973).

[7] K Schwarzschild: ber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. Sitzungsberichte der Kniglich Preussischen Akademie der Wissenschaften 7: 189196 (1916). For a translation, see S Antoci and A Loinger: On the gravitational field of a mass point according to Einstein's theory. arXiv:physics/9905030 [physics].(1999).

[8] E Noether: Mathematisch-Physikalische Klasse 2,235 (1918). English translation in Transport Theory and Statistical Physics. 1(3), 186 (1971).

[9] A H Kara and F M Mahomed: Relationship Between Symmetries and Conservation Laws. Int. J. Theor. Phys. 39,23-40 (2000).

[10] N H Ibragimov and T Koslurd: Lagrangian Approach to Evolution Equations, Symmetries and Conservation Laws. Nonlin. Dynam. 36(1), 29-40 (2004).

[11] N H Ibragimov, A H Kara and F M Mahomed: Lie Backlund and Noether symmetries with applications. Nonlin. Dynam., 15, 115136 (1998).

[12] G W Bluman and J D Cole: Similarity Methods for Differential Equations. Springer-Verlag, New York (1989).

[13] G W Bluman and S Kumei: Symmetries and Differential Equations. Springer-Verlag, New York (1989).

[14] N Euler and W H Steeb: Continuous symmetries, Lie Algebras and Differential Equations. Bibligraphisches Institut, Mannheim (1992).

[15] P E Hydon: Symmetry Methods for Differential Equations. Cambridge University Press, Cambridge (2000).

[16] N H Ibragimov: CRC Handbook of Lie Group Analysis of Differential Equations, Vol 1, Symmetries, Exact Solutions and Conservation Laws. CRC Press, Boca Raton (1994).

[17] N H Ibragimov: CRC Handbook of Lie Group Analysis of Differential Equations, Vol 2, Applications in Engineering and Physical Sciences. CRC Press, Boca Raton (1995).

[18] N H Ibragimov: CRC Handbook of Lie Group Analysis of Differential Equations, Vol 3, New Trends in Theoretical Developments and Computational Methods. CRC Press, Boca Raton (1996).

[19] P J Olver: Applications of Lie Groups to Differential Equations. Springer-Verlag, New York (1986).

[20] L V Ovsiannikov: Group Analysis of Differential Equations. Academic Press, New York (1982).

[21] N H Ibragimov: Elementary Lie Group Analysis and Ordinary Differential Equations. John Wiley & Sons, Chichester (1999).

[22] H Stephani: Differential Equations: Their Solution Using Symmetries. Cambridge University Press, Cambridge (1989).

[23] G Griffiths and W Schiesser: Linear and nonlinear waves, Scholarpedia, the peer-reviewed open-access encyclopedia, 4(7):4308 (2009).

[24] G Robert and A Robert: Water Wave Mechanics for Engineers & Scientists. Advanced Series on Ocean Engineering-Vol2, World Scientific, (2007).

[25] A Yariv and P Yeh: Optical Waves in Crystals: Propagation and Control of Laser Radiation. Volume 54 of Wiley Series in Pure and Applied Optics (2002).

[26] L Eyges: The Classical Electromagnetic Field. New York: Dover Publications Inc (1972).

[27] S Farlow: Partial Differential Equations for Scientists and Engineers. Chapter 17, Dover Publications, New York (1993).

[28] M Ablowitz, D Kaup, A Newell and H Segur: Method for Solving the Sine-Gordon Equation. Phy. Rev. Lett. 30,1262 (1973).

[29] R Abraham and J Marsden: Foundations of Mechanics. Benjamin/Cummings, (1978).

[30] Fornberg, Bengt, and G B Whitham: A numerical and Theoretical Study of Certain Nonlinear Wave Phenomena. Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 289.1361, 373-404, (1978).

[31] Wineberg, B Stephen, McGrath, F Joseph, Gabl, F Edward, L Ridgway Scott, Southwell and E Charles: Implicit Spectral Methods for Wave Propogation Problems. Journal of Computational Physics, Volume 97, Issue 2, p. 311-336, (1991).

[32] M Toda: Nonlinear Waves and Solitons. Kluwer, (1989).

[33] B Vick and R L West: Analysis of Damped Waves Using the Boundary Element Method. Wet Transaction. Model.Simulation. 15 265-278 (1997).

[34] G Todorova and B Yordanov: Critical Exponent for a Nonlinear Wave Equation with Damping. J. Diff. Eq. 174 464-489 (2001).

[35] G Todorova and B Yordanov: Nonlinear Dissipative Wave Equations with Potential. AMS Cont. Math. 426 317-337 (2007).

[36] J Wirth: Wave Equations with Time-dependent Dissipation II, Effective dissipation. J. Diff. Eq. 232 74-103 (2007).

[37] M A Rincon and M I Copetti: Numerical Analysis for a Locally Damped Wave Equation. J. Appl. Anal. Comp. Volume 3, Number 2, 169-182 (2013).

[38] Mouhamad Jradeh and Maitine Bergounioux: On Solutions of a Non linear Wave Equation Derived From Brain Activity Modeling. preprint (2008).

[39] Y M Ali and L C Zhang: Relativistic Heat Conduction. International Journal of Heat and Mass Transfer 48, 2397-2406 (2005).

[40] Michiel Hazewinkel: D'Alembert operator. Encyclopedia of Mathematics, Springer, ISBN 978-1-55608-010-4, ed.(2001).

[41] H Stephani, D Kramer, M Maccallum, C Hoenselaers and E Herlt: Exact Solutions of Einstein's Field Equations (Second Edition). Cambridge University Press (2003).

[42] A H Bokhari and A Qadir. J. Math. Phys. 28, 1019 (1987).

[43] M Z Bashir and S Ehsan. Il Nuovo Cimento B 123, 1 (2008).

[44] M J Amir, A H Bokhari, A J Qadir. Math. Phys. 35, 3005 (1994).

[45] F Ali and T Feroze: Classification of Plane Symmetric Static Space-Times According to Their Noether Symmetries. Int J Theor Phys 52:3329-3342 (2013).

[46] T Feroze, A Qadir and M Ziad: The Classification of Plane Symmetric Spacetimes by Isometries. J.math. phys. vol42, number 10 (2001).

[47] M F Shamir, A Jhangeer and A A Bhatti: Killing and Noether Symmetries of Plane Symmetric Spacetime. Int. J. Theor. Phys. 52(9): 3106-3117 (2013).

[48] W F Ames, R J Lohner and E Adams: Group Properties of $u_{tt} = (f(u)u_x)x$, Internat. J. Non-Linear Mech. 16, 439-447 (1981).

[49] V A Baikov, R K Gazizov and N H Ibragimov: Classification According to Exact and Approximate Symmetries of Multidimensional Wave Equations. Akad. Nauk SSSR Inst. Prikl. Mat. Preprint no. 51(1990).

[50] G Bluman and S Kumei: On Invariance Properties of the Wave Equation. J. Math. Phys. 28, 307-318 (1987).

[51] M L Gandarias, M Torrisi and A Valenti: Symmetry Classification and Optimal Systems of a Non-linear Wave Equation. Internat. J. Non-Linear Mech. 39, 389-398 (2004).

[52] A Oron and P Rosenau: Some Symmetries of Nonlinear Heat and Wave Equations. Phys. Lett. A 118 (4), 172-176 (1986) .

[53] H Azad and M T Mustafa: Symmetry Analysis of Wave Equation on Sphere. J. Math. Anal. Appl. 333, 1180-1188 (2007).

[54] Joseph Silk: Horizons of Cosmology. Templeton Press. p. 208 (2009).

[55] Alexander Friedmann: ber die Mglichkeit einer Welt mit konstanter negativer Krmmung des Raumes. Zeitschrift fr Physik A 21 (1): 326-332 (1924), English trans. in 'General Relativity and Gravitation' vol.31 (1999).

[56] B Abbasi and W Craig: Proc. R. Soc. A 470,(2014) 2014036.

[57] A H Bokhari, F M Mahomed and F D Zaman: Journal of Mathematical Physics, Vol. 53 (2012) 043703.

[58] E R G Eckert and R M Drake: Analysis of Heat and Mass Transfer, McGraw-Hill Kogakusha, Tokyo, 1972.

[59] P M Morse and H Feshback: Methods of Theoretical Physics, McGraw-Hill, New York, 1953.

[60] C R Cattaneo: Sur une de l'equation de la chaleur eliminant le paradoxe d'une propagation instantanee, Compte. Rend. 247 (4) (1958) 431433.

[61] P Vernotte: Les paradoxes de la theorie continue de l'equation de la chaleur, Compte. Rend. 246 (22) (1958) 31543155.

[62] C Bai and A S Lavine: On Hyperbolic Heat Conduction and the Second Law of Thermodynamics, J. Heat Transfer, Trans. ASME 117 (2) (1995) 256263.

[63] A Barletta and E Zanchini: Hyperbolic Heat Conduction and Local Equilibrium: a second law analysis, Int. J. Heat Mass Transfer 40 (5) (1997) 10071016.

[64] M B Rubin: Hyperbolic Heat Conduction and the Second Law, Int. J. Eng. Sci. 30 (11) (1992) 16651676.

[65] Y M Ali and L C Zhang: Relativistic Moving Heat Source, Int. J. heat and mass transfer, Vol. 48 (2005) 2741.

[66] Y Mambrini: Histories of Dark Matter in the Universe, Universite paris-sud, F-91405 Orsay, France.

Vitae

- Name: Usamah Al-Ali
- Nationality: Saudi
- Date of Birth: 1-7-1973
- Email: *usamah1973@hotmail.com*
- Permenant Address: Ministry of Education, Al-Hasa-Saudi Arabia

Usamah Sadeg Al-Ali was born on the first of July, 1973 in Al-Hasa, Saudi Arabia. He joined King Fahd University of Petroleum and Minerals in 1991. He obtained his BS degree in Mathematics in 1996. He worked for the Saudi ministry of education since 1996. He rejoined KFUPM in 2009 for conducting higher studies in Mathematics where he obtained his master degree in 2012. His field of specialization is applied math. In particular, his research interests are focusing on the application of symmetry analysis and conservation laws to equations arising in mathematical Physics and cosmology. In this context, he has published the following papers in reputed scientific journals:

1 - Usamah S. Al-Ali, Ashfaque H. Bokhari, A. H. Kara and F. D. Zaman, " On the Symmetries and Conservation Laws of the Multidimensional Nonlinear Damped Wave Equations", Advances in Mathematical Physics Volume 2017, Article ID 9401205, 11 pages, https://doi.org/10.1155/2017/9401205.

2 - Usamah S. Al-Ali, Ashfaque H. Bokhari, A. H. Kara and F. D. Zaman," Symmetry Analysis and Exact Solutions of the Damped Wave Equations on the Surface of the Sphere ", Advances in Differential Equations and Control Processes, Pushpa Publishing House, Allahabad, India Published: November 2016, http://dx.doi.org/10.17654/DE017040321 Volume 17, Number 4, 2016, Pages 321-333.

3- Usamah S. Al-Ali, Ashfaque H. Bokhari, A. H. Kara and F. D. Zaman, " Invariance Properties and Conservation Laws of the Nonlinear Damped Wave Equation with Power Law Nonlinearities ", Results in Physics (2017), doi: http://dx.doi.org/10.1016/j.rinp.2017.02.026.